DETERMINATION OF THE CAPILLARY SURFACE AND EIGEN-ANALYSIS OF ITS OSCILLATIONS UNDER LOW-GRAVITY : A MATHEMATICO-NUMERICAL (F. E. M.) STUDY

by

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CERTIFICATE

It is certified that the work contained in the thesis entitled "DETERMINATION OF THE CAPILLARY SURFACE AND EIGEN-ANALYSIS OF ITS OSCILLATIONS UNDER LOW-GRAVITY. A MATHEMATICO - NUMERICAL (F.E.M.) STUDY", by S. Saha has been carried out under our supervision [and that this work has not been submitted elsewhere for a degree.]

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The study of the static configuration of capillary surfaces as well as their dynamic behaviour has wide applications in many technological areas especially in space technology. The capillary surface is governed by the following "Laplace-Young" equation [Myskis A.D. : Low Gravity Fluid Mechanics (Springer-Verlag) 1987.]

$$\left.\begin{aligned} -\underline{\nabla}\left[\frac{\underline{\nabla}u}{\sqrt{1+|\underline{\nabla}u|^2}}\right] + bou &= c \quad \text{in } \Omega \\ \frac{\underline{\nabla}u.n}{\sqrt{1+|\underline{\nabla}u|^2}} &= \cos\theta_c \quad \text{on } \partial\Omega \end{aligned}\right\} \quad \text{(P-1)}$$

When Ω is known a-priori the problem amounts to the solution of a system of non-linear equations. In view of the early development there has been an extensive study even in standard monograph form

dealing with the question of the existence of solution of (P-1) [Finn R. : Equilibrium Capillary Surfaces (Springer-Verlag) 1986].

The present thesis concerns itself with the computational aspects of the solution of (P-1) and eigen-analysis of oscillations of an ideal liquid contained in rigid containers under low-gravity conditions.

(1) Generally Ω is not known a priori and therefore the problem becomes a "free-boundary" problem. In such a case the problem of the computation of the capillary surface with volume constraint on the liquid becomes quite complicated. In the thesis the above problem is treated invoking shape-optimization formalism and a mathematical justification of the method is derived. This seems to be the first attempt to solve the problem in this manner.

The basic idea behind the application of this formalism consists of the following:

For an admissible set M determine $\Omega^* \in M$ s.t.

$J(\Omega^*) \leq J(\Omega) \;\forall\; \Omega \in M$

Where $J(\Omega)$ is the energy functional for Ω. Here Ω corresponds to a shape of the domain and it is varied over an admissible set M. We derive a mathematically justified numerical procedure based on ideas outlined in the work of Pironneau [Pironneau O : Optimal Shape Design for Elliptic Systems. (Springer-Verlag) 1984].

(2) Earlier numerical simulation studies using finite-elements have been carried out for (P-1) when Ω is known a priori [Brown R.A. :[Finite - Element Methods for the calculation of Capillary

Surfaces. Jour of Comp. Phys. 33 (1979)]

In this situation also, inspite of the presence of high gradients in certain regions no adaptive scheme was available. Hence possibly for the first time a reliable adaptive strategy for the h-version finite-elements has been formulated. It is based on some error indicators suitable for this problem in particular.

(3) Then we consider the problem related to the study of oscillations of an ideal liquid in a rigid container. The equation governing the liquid oscillations are as follows (Myskis A.D. et al.: Low-Gravity Fluid Mechanics. (Springer- Verlag) 1987].

$$\Delta\phi = 0 \quad \text{in } \Omega$$

$$\frac{\partial\phi}{\partial \underline{n}} = 0 \quad \text{in } \Sigma$$

$$-\Delta_{\Gamma}\left[\frac{\partial\phi}{\partial \underline{n}}\right] + bo\left[\frac{\partial\phi}{\partial \underline{n}}\right] + C = w^2\phi \qquad \text{in } \Gamma$$

$$\frac{\partial}{\partial e}\left[\frac{\partial\phi}{\partial \underline{n}}\right] + X \quad \frac{\partial\phi}{\partial \underline{n}} = 0 \qquad \text{on } \partial\Gamma$$

$$\int \frac{\partial\phi}{\partial \underline{n}} = 0 \text{ [volume constraint condition]}$$

There has been quite an amount of work in this direction using semi-analytic methods [Bauer H.F. : Linear Liquid Oscillations in Cylindrical Container under Zero-Gravity. Technical Report. Universität Bundeswehr München (1989)].

However the range of applicability of these methods is very

limited. Therefore we propose a finite-element method for the calculation of eigen-values based on the variational formulation. The most distinguishing feature of the above problem is the presence of higher-order differential operator on the boundary surface. Therefore it does not allow standard procedure to reduce it to a variational form. However this has been achieved by introducing two distinct operators which together help in reducing the problem to a standard eigen-value problem.

(4) The problem of computation of eigen-values for vibration of liquid in a container containing a large number of rigid tubes is very important form the point of applications. We deal with this problem using asymptotic methods based on homogenization principle. The problem is fairly complicated due to its 3-dimensional nature. For solving this problem we make use of asymptotic analysis in the spirit of Lions [Lions J.L : Some Methods in the Mathematical Analysis of Systems and their Control (Gordon Breach) 1981] and construct the homogenized operators.

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LIST OF SYMBOLS

Ω	Bounded open set in $\mathbb{R}^n$
$C^m(\Omega)$	Space in m times continuously differentiable functions
$C^\infty(\Omega)$	Space of infinitely differentiable functions in Ω
$W^m_p(\Omega)$	Sobolev Space of order m.
$H^m(\Omega)$	Sobolev space $W^m_2(\Omega)$
Ω_l	Domain occupied by the liquid
Ω_g	Domain occupied by the gas
σ_l	Surface tension of the liquid
σ_g	Surface tension of the gas-solid interface
σ_s	Surface tension of the liquid-solid interface
ρ_l	Density of the liquid
ρ_g	Density of the gas
g	Acceleration due to gravity
bo	Parameter $\rho_l g/\sigma_l$
Σ_l	Surface of contact of the liquid with the solid
Σ_g	Surface of contact of the gas and the solid
s	Free surface of the liquid
$\underline{\nabla}$	Grad operator
$\underline{\nabla}_\Gamma$	Beltrami Differential operator
N^i	Finite-Element basis function at node i
q^k	Vector of the coordinates of node k
q	Arbitrary constant corresponding to the Lagrangian multiplier for the volume constraint of the liquid
r_h	Residual Error
e_h	Solution Error
$P_p(K)$	Space of polynomials of order $p > 1$ on K
$V_h(\Omega)$	Finite-Element spaces defined in Chapter IV

$V_{ho}(\Omega)$	Finite-Element spaces defined in Chapter IV
X	Characteristic function
γ	Trace operator
A, B	Stiffness and Mass operators defined in Chapter V
M_{α}	Symmetrized Added-Mass operator
λ	Eigen-value of the free-surface oscillations
ω	Circular frequencies of oscillations

CHAPTER I
INTRODUCTION

1.1 MOTIVATION :

With the development of space-technology and the diversification of investigations conducted , the role of the fluid is becoming very important. The space-vehicles contain liquid fuel tanks and many other liquid containing devices. Various types of fluids are used in the rocket engines power units, life support systems, in the temperature control equipments etc. The static fluid configurations can provide essential information for storage, capillary pumping, venting etc. whereas the dynamic analysis of a liquid is very important for the design of the structural systems and more importantly for the stability of the system. Hence, the design and operation of such systems calls for the theoretical and experimental investigations of the behaviour of the liquid under low-gravity conditions. However, conducting experiments in space is expensive whereas the simulation of zero-gravity under terrestrial conditions is complicated and expensive. Thus there is a growing need for a rigorous mathematical analysis along with computational studies in order to produce substantial results so that the fluid behaviour may be predicted under extreme conditions.

There is a special interest to study the static and dynamic behaviour of the liquids in a vessel under low gravity conditions. The present study has been aimed at liquids in rigid containers and shall not deal with problems of liquid drops, bubbles, liquid bridges etc. which may be important from the practical point of view. As the natural frequencies of oscillations of the liquid are much lower than the structural

frequencies the vessel is considered to be rigid.

The basic problem concerning the static state is the determination of the capillary surface of equilibrium. The capillary surface is governed by the following "Laplace-Young" equation.

$$\left.\begin{aligned} -\underline{\nabla}\left[\frac{\underline{\nabla} u}{\sqrt{1+|\underline{\nabla} u|^2}}\right] + bo\, u = C \quad \text{in } \Omega \\ \frac{\underline{\nabla} u \cdot \underline{n}}{\sqrt{1+|\underline{\nabla} u|^2}} = \cos\theta_c \quad \text{on } \partial\Omega \end{aligned}\right\} \qquad \text{(P-1)}$$

where u stands for the surface, bo is a system parameter depending on the gravity, θ_c is the angle of contact (for details refer to Chapter IV)

If H denotes the mean curvature of the surface

[Nitsche (1976), Myskis (1987)], then $2H = \underline{\nabla}\left[\frac{\underline{\nabla} u}{\sqrt{1+|\nabla u|^2}}\right]$

Then it can be seen that $2H = C - bou$ (1.1)

is obtained from (P-1). Since the mean curvature can be explicitly expressed in terms of a function of the space variables, the capillary surface is often described as a surface of prescribed mean curvature with a given contact angle on the boundary. In the case when the gravity is zero one has $bo = 0$ and one can find from (1.1) that H is constant. Thus the capillary surface is termed as a surface of constant mean curvature or the so-called H surfaces. If in addition there is no volume constraint [ie. $C = 0$] one arrives at the problem of minimal

surfaces or the Plateau's problem. The problem in its generality is to solve (P-1) when Ω is not known a priori. This is termed in literature as the "free-boundary" problem whose numerical solution is in general difficult to obtain. Possibly the first published attempt to numerically solve the 3-D meniscus is by Orr et al [1977]. They adopted a direct method to solve (P-1) using finite-element method. The above work was essentially a simulation study without any attempt to justify the method. Under these circumstances there is a need to develop a numerical scheme with a proper mathematical basis and to show the practical viability of the method.

In earlier numerical simulation studies finite-element method has been applied by some authors to solve(P-1) when Ω is known a priori. In view of the presence of high gradients of u in certain regions of liquid container domain there is a need for the improvement in the methodology. In other words, one has to change the model, mesh or the approximating structure of the computational methods so as to improve the quality of the solution. This is usually termed as adaptive refinement. Unfortunately, there were no adaptive schemes available for the problem. Thus, there is a need to develop a practical adaptive scheme for the problem. In this connection it is to be noted that as the operator occuring in (P-1) is non-linear, it is a non-trival task to obtain a reliable criteria for adaptive refinement. The study of the oscillations of the liquid is complicated from theoretical and computational point of view. The equations governing the oscillations of an ideal liquid are as follows :

$$\left.\begin{aligned}
&\Delta\phi = 0 \quad \text{in } \Omega \\
&\frac{\partial\phi}{\partial \underline{n}} = 0 \quad \text{in } \Sigma \\
&-\Delta_\Gamma\left[\frac{\partial\phi}{\partial \underline{n}}\right] + bo\left[\frac{\partial\phi}{\partial \underline{n}}\right] + C = \omega^2\phi \qquad \text{on } \Gamma \\
&\frac{\partial}{\partial \underline{e}}\left[\frac{\partial\phi}{\partial \underline{n}}\right] + X\frac{\partial\phi}{\partial \underline{n}} = 0 \qquad \text{on } \partial\Gamma
\end{aligned}\right\} \quad \text{(P-2)}$$

where ϕ stands for the velocity potential of the fluid, ω the square root of the eigen value, C a constant corresponding to the Lagrangian multiplier for the volume constraint, n the outer normal and e the tangent vector to the surface at the line of contact and X a system parameter defined on the line of contact. (For details refer to Chapter V).

Earlier some computational studies have been carried out by Bauer [1963a, 1963b, 1981, 1981, 1989a, 1989b, 1989c] using semi-analytic methods. The method is not suitable for complex geometries or even problems of higher dimensions. For some Soviet works reference can be made to Myskis [1987]. The finite - element method based on variational formulation can serve as an effective method to overcome these difficulties. One basic difficulty in the application of this method arises from the fact that the differential operator appearing on the boundary Γ is of higher order than the differential operator in the interior of the domain (Ω). Thus the present structure of (P-2) is not readily amenable to a simple application of finite element method. Thus there is a necessity to develop a procedure in order to reduce (P-2) to a

standard eigen-value problem in the variational formulation.

The computation of the eigen-frequencies of the liquid oscillations containing tube bundles has important applications. They are useful in the study of reactors where the number of tubes is of the order of several tens of thousands. A substantial amount of work has been devoted to the computational aspects of the study of vibration of the tube-bundles by Planchard [1980, 1982, 1983]. Planchard [1980,1983] applied the theory of homogenization to compute the eigen-frequencies. In low-gravity environment the capillary surface oscillations become important as the surface-tension assumes a dominating role. The problem is 3-dimensional in nature and homogenization is not a simple task requiring the construction some special operators.

1.2 NATURE AND OUTLINE OF THE WORK

Keeping in view of the fact that the capillary problem is an old one and some amount of work has been done, it has been proposed to address the following four different aspects of the capillary surface problem.

1) The problem of construction of the capillary surface using shape optimization technique .
2) The problem of adaptive refinement scheme for the Finite Element Method .
3) The problem of the computation of the eigen-frequencies of oscillations of an ideal liquid under low-gravity .
4) The problem of the computation of the eigen-frequencies of an ideal liquid containing a rigid tube-bundle using the theory of homogenization.

The above problems are presented along with the state of the art, methodology, implementation and results in chapter III, IV, V and VI repectively. In addition, the mathematical preliminaries are given in chapter II and overall conclusions in chapter VII.

The present work develops numerical schemes giving proper mathematical justification. Adequate numerical computations have been carried out to support the practical viability of the methods. The software has been developed in FORTRAN language and the computations have been carried out on H.P. 9000/850 Mainframe computer and Graphic workstations.

The contents of Chapter III and IV have been accepted for publication in the Journal "Numerische Mathematik" and Journal "Computer Methods in Applied Mechanics and Engineering" respectively, under the following titles :

1. "A Shape-Optimization Technique for the Capillary Surface Problem".
2. "A Finite-Element Adaptive Strategy for the Capillary Surface Problem."

   The contents of Chapter V have been submitted to Journal "Computer Methods in Applied Mechanics and Engineering" under the following title.
3. "A Numerical Method (using Finite-Elements) for the Computation of the Eigen-frequencies of Oscillations of an Ideal Liquid in Low Gravity."

CHAPTER II

MATHEMATICAL PRELIMINARIES

The aim of this section is to present some important classical results and definitions which will be useful in the sequel.

We shall denote Ω as an open set in $\mathbb{R}^n$ with boundary $\partial\Omega$. Then we have the following definitions :

$C^m(\Omega)$ denotes the family of functions with domain Ω whose m^{th} derivative is continuous.

$C^\infty(\Omega)$ denotes the family of infinitely differentiable functions.

$D(\Omega)$ denotes the sub-set of $C^\infty(\Omega)$ which have compact support in Ω.

We denote $L^p(\Omega)$ for $1 \le p < \infty$ as the space of equivalence classes of functions s.t. $\|f\|_{L^p} = \int|f|^p < \infty$. [The symbol $\int$ (.) shall be used often instead of the complete notation $\int(.)\, d\Omega$ for simplification purposes and the integration is understood to be in the Lebesgue sense]

When $p = \infty$, $L^\infty(\Omega)$ denotes the space of equivalence classes of functions which are bounded almost everywhere (a.e.). It is a Banach space with the norm defined by $\|f\| = \operatorname*{ess\,sup}_{\underline{x} \in \Omega} \|f(\underline{x})\|$

For $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have the Hölder inequality

$$\left|\int_\Omega fg\right| \le \|f\|_{L^p} \|g\|_{L^p} \quad \forall\, f \in L^p(\Omega) \text{ and } g \in L^q(\Omega) .$$

The multi-index α is defined as the n-tuple

$$\alpha = (\alpha_1, \alpha_2, \alpha_n) \ge 0 \quad \alpha_i \text{ are integers}$$

Associated with the multi-index we have the following symbols

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

$$\underline{x}^{\alpha} = x_1^{\alpha_1} \dots . x_n^{\alpha_n} \quad \underline{x} \in \mathbb{R}^n \quad .$$

We say that two multi-indices α and β are related by $\alpha \le \beta$ if $\alpha_i \le \beta_i \ \forall \ 1 \le 1 \le n$.

Finally we express the differential operator through the multi-index.

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \quad .$$

With the help of the above notations we can now define the Sobolev space $W_p^m(\Omega)$ as the following

$$W_p^m(\Omega) = \left\{ u \in L^p(\Omega) \ : \ D^{\alpha}u \in L^p(\Omega) \ \forall \ |\alpha| \le m \right\}$$

equipped with the norm

$$\|u\|_{m,p,\Omega} = \Big(\sum_{|\alpha| \le m} \|D^{\alpha}\|_{L^p(\Omega)} \Big)^{1/p}$$

When $p=2$ the space $W_2^m(\Omega)$ will be denoted by $H^m(\Omega)$.

and for any $u \in H^m(\Omega)$ we denote its norm by $\|u\|_{m,\Omega}$ i.e.

$$\|u\|_{m,\Omega} = \|u\|_{m,2,\Omega} \quad .$$

An extension operator P_{Ω} for $W_p^m(\Omega)$ is a bounded linear operator

$$P_\Omega : W_p^m(\Omega) \to W_p^m(\mathbb{R}^n)$$

s.t $P_\Omega u|_\Omega = u \; \forall \; u \in W_p^m(\Omega)$

The norm of P_Ω will depend on Ω in general. Next we shall define a class T of domains for which the norms of corresponding extensions do not depend on $\Omega \in T$. In such a case we say that $\Omega \in T$ has the so-called uniform extension property.

This extension property is summed up in the following theorem due to Chenais [1975]

Theorem 2.1 :-

Let θ, h, r be three real numbers $(\theta \in (0,\pi/2), 2r<h)$ and $m \in \mathbb{N}$(set of +ve integers). $\exists$ a +ve constant $K(\theta,h,r)$ depending on $\Omega \in T(\theta,h,r)$ only through θ,h,r, and such that $\forall \Omega \in T(\theta,h,r)$ $P_\Omega : H^m(\Omega) \to H^m(\mathbb{R}^n)$

is a linear and continuous extension operator, s.t.

$$\|P_\Omega\| < K(\theta,h,r) .$$

We shall now state the Rellich-Kondrasov compactness theorem which has got important applications.

Theorem 2.2 (Rellich-Kondrasov) : -

Let $\Omega \subset \mathbb{R}^m$ be a bounded open set of Class C^1. Then the following inclusions are compact.

(i) if $p<m$, $W_p^m(\Omega) \to L^q(\Omega)$ $1 \le q < p^*$ [where $p^* = \frac{mp}{m-p}$]

(ii) if $p=n$, $W_n^m(\Omega) \to L^q(\Omega)$ $1 \le q < \infty$

(iii) if $p>m$, $W_p^m(\Omega) \to C(\bar{\Omega})$.

The notion of trace has very important applications in the boundary value problems. The trace is a generalization of the concept of the restriction of a continuous function to a submanifold of its domain of definition. We shall now state the Trace theorem as follows.

Theorem 2.3 (The Trace Theorem for $H^m(\Omega)$).

Let Ω be a smooth bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ be its boundary. Then the trace operators γ_k, $0 \le k \le m-1$, can be extended to continuous linear operators mapping $H^m(\Omega)$ onto $H^{m-k-1/2}(\partial\Omega)$. Further the operator $\gamma = (\gamma_0, \gamma_1 \cdots, \gamma_{m-1})$ is a continuous linear mapping of $H^m(\Omega)$ onto $\Pi\, H^{m-k-1/2}(\partial\Omega)$. Moreover $\exists$ a continuous linear (right) inverse γ^{-1} mapping $\Pi_{k=0}^{m-1}(\partial\Omega)$ into $H^m(\Omega)$.

The details about Sobolev Spaces can be found in the text books of Adams [1975], Oden and Reddy [1976] and Mason [1985]. For the general concepts in functional analysis one may refer to Yosida [1974], Taylor [1958] or any standard text books in functional analysis.

CHAPTER III

A SHAPE-OPTIMIZATION TECHNIQUE FOR THE CAPILLARY SURFACE PROBLEM

3.1 : INTRODUCTION

The study proposes a method to construct the capillary surface, based on the shape-optimization technique. The problem of determination of capillary surface is reformulated as an optimization problem over an admissible class of sets. An existence theorem for the optimal solution is proved and a numerical scheme is proposed based on the finite element method and the gradient method where the gradients are calculated with respect to the shape of the domain. Some model problems have been solved.

3.2 : LITERATURE SURVEY AND GENERAL BACKGROUND

Whenever a liquid and a gas remain adjacent to each other without mixing, the interface between them is termed as the capillary surface. Although the capillary phenomenon was observed even in the times of antiquity a consistent theory capable of scientific predictions was first proposed by Young [1805]. He introduced the notion of mean-curvature of the surface and could relate it to the surface tension. He furthermore observed the constancy of the angle of contact of a liquid surface with a solid. Laplace [1806] derived practically the same results as Young. The theory was further strengthened when Gauss [1830] arrived at the same equations using the theory of virtual work.

The equation governing the capillary surface is the "Laplace-Young" equation which is a non-linear elliptic equation. The general solution becomes particularly difficult not only because the equation is non-linear but also the domain of definition is not known a priori. This is termed in literature as the "free-boundary" problem whose numerical solution is in general difficult to obtain. The basic problem in which the mathematical study is aimed at is the problem of existence (or non-existence) of the solution surface under various conditions. Perhaps the first published work of this nature is by Concus and Finn [1965]. The paper discusses the existence of the surface in a domain with corners and the discontinuous dependence of the solution on the boundary which was a significant contribution. The discontinuous dependence has also been verified experimentally. Similar mathematical works on the existence were done by Concus and Finn [1974a,b,c, 1976], Finn [1974, 1979, 1983, 1986, 1988], Giaquinta [1974], Guisti [1978,1980]. A comprehensive account of the mathematical theory is given in the monograph by Finn [1986]. Though there are no generalized numerical schemes to solve capillary surface problem with volume constraints, in several practical situations some simplifications are possible. One such simplification is when the domain of definition is known such as when the liquid is contained in a cylindrical vessel and gravity acts along the generatrix.

The construction of the capillary surface by numerical methods was studied by Concus [1968], and later by Siekmann, Scheideler and Tietze [1981a,b]. They used traditional methods such as expansion matching along with the finite-difference

methods. Since for fixed domains variational formulation is possible, several researchers used variational Finite Element Method which has the advantage of handling complex domain geometries easily. Such an approach has been used by Orr [1975,1977], Brown [1979] and Mittelmann [1977] for the capillary surface problem. Brown [1979] used Galerkin Finite Element Method for solving the capillary surface problem of a liquid in a vessel of square cross-section. He compared the efficiencies of various finite-element methods with different interpolation functions. Mittelmann [1977] derived some a priori error estimates for the Finite Element Method for the capillary problem.

Another class of problems that are simple to solve are the axi-symmetric problems [e.g. spherical and conical vessels etc]. In these case although the domain is not known a-priori, the dimensions of the problem can be reduced and hence the problem becomes simplified. A number of studies were carried out for this class of problems [Bashforth and Adams [1883], Chesters [1977], Padday and Pitt [1972,1973]]. Although it is not possible to obtain a closed form solution when gravity is small, one can solve the equation explicitly for the zero-gravity case. The solution is expressed in the form of elliptic integrals and it can be represented in terms of Delaunay curves. However, use of the volume constraint poses some serious difficulties, as the volume is a quantity which has to be determined numerically and one does not know all the parameters in advance. Hence the explicit solution method cannot be used and it is a good choice to solve the boundary-value problem by means of shooting techniques. This technique was adopted by Bashforth and Adams [1883] which was

possibly the first time when fairly comprehensive results were published and since then the work has become a bibliographical rarity. Axi-symmetric profiles for ranges beyond the stability regions were computed by Padday and Pitt [1972,1973]. A large amount of numerical data on axi-symmetric equilibrium shapes is contained in the monograph by Hartland and Hartley [1976].

According to the literature available the numerical solution of 3-Dimensional meniscus problem was reported for the first time by Orr et al. [1977]. In their work Orr et al. used finite element method for the solution of the capillary surface equation. Their work was basically a simulation study without any attempt to justify the method. Under these circumstance there is a need to develop a numerical scheme with a proper mathematical basis and practical viability.

The present work adopted a different strategy to solve the free-boundary problem. The potential energy of the system is evaluated as a function of domain and is minimized by changing the shape of the domain occupied by the liquid. Hence, it becomes a "shape-optimization" problem. For the numerical scheme the problem is solved using finite element method and a gradient based method. The gradient of the functional is calculated with respect to the shape of the domain defined by the nodes of the finite-element mesh [Pironneau [1984]]. It is worthwhile to mention that the Augmented Skeleton Method (ASM) developed by Hornung and Mittelmann [1989a,1989b,1990] is fairly a generalized method to solve the capillary surface problem which takes into account the volume constrint and the surface need not be representible as the graph of a function. Though the A.S.M is a

general method it is felt that the proposed method is more efficient for the class of problems in which the surfaces are representible by the graph of a function.

3.3 : NOTATIONS AND PRELIMINARIES :

Let us consider a mechanical system comprising of liquid, gas and solid as depicted in Fig. 3.1. The liquid fills the vessel partially and the gas or the vapour of the liquid fills the remaining portion of the vessel. The walls of the vessel are considered to be rigid and sufficiently smooth.

Let D denote the total region of the vessel Ω_l the region occupied by the liquid and the region occupied by gas being denoted by Ω_g. The liquid has a specified volume vol. That is the volume of D is finite and larger than the volume of the liquid in order to make certain that the liquid remains strictly inside the vessel. Let Σ_l and Σ_g represent the surfaces of the vessel in contact with the liquid and with the gas respectively. The interface between the liquid and the gas is denoted by S and γ denotes the line of contact between the surface of the vessel and the surface of the liquid (Fig. 3.1) a point on which is denoted by P.

Let ρ_l denote the density of the liquid, the density of the gas is small and is neglected. The surface-tensions of the liquid, liquid-solid and the solid-gas interfaces are denoted by σ_l, σ_s and σ_g respectively.

Let us denote $(\sigma_g-\sigma_s)/\sigma_l$ by $\cos\theta_c$. θ_c denotes the angle of contact which is the dihedral angle formed by the liquid and the vessel on the line of contact. The liquid, the gas and the vessel material are assumed to be homogeneous.

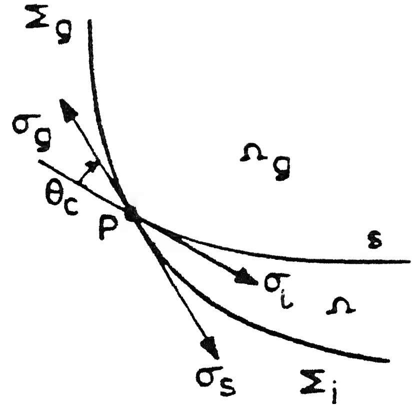


FIG. 3.1 NOTATIONS

The external forces acting on the liquid are only due to gravity and assumed to be independent of time. Thus the volume density of these forces is given by $-\rho\nabla\bar{\pi}$, where $\bar{\pi}$ is the gravitational potential.

Thus, the total potential energy of the system can be expressed by the following

$$U = \sigma_l |S| + \sigma_s |\Sigma_l| + \sigma_g |\Sigma_g| + \rho_l \int_{\Omega_l} \bar{\pi} \, d\Omega_l \tag{3.1}$$

[where $|.|$ denotes the measure of the corresponding surface].

Fig. 3.2 shows the representations of the system under consideration x_1, x_2 and z are the Cartesian Coordinates and g is the acceleration due to gravity directed as shown. The vessel lies entirely in the region $z \geq 0$. Ω is the region formed by the projection S onto the plane $z = 0$, and Γ denotes the boundary of Ω.

Let Ω_d denote the region formed by the projection of the vessel surface onto the plane $z = 0$, and Γ_d the boundary of Ω_d. The surface 'S' is expressed as a single-valued function $u(\underset{\sim}{x} \in \Omega)$. The vessel is assumed to be open and its surface represented by a single-valued function $\psi(\underset{\sim}{x})$ $(\underset{\sim}{x} \in \bar{\Omega}_d)$ and $\psi \in C^1(\bar{\Omega}_d)$.

Hence we have for the system

$$u - \psi > 0 \ \forall \ \underset{\sim}{x} \in \Omega \tag{3.2}$$

(from the conditio' that the surface of the liquid lies entirely within the vessel).

From the volume condition we have

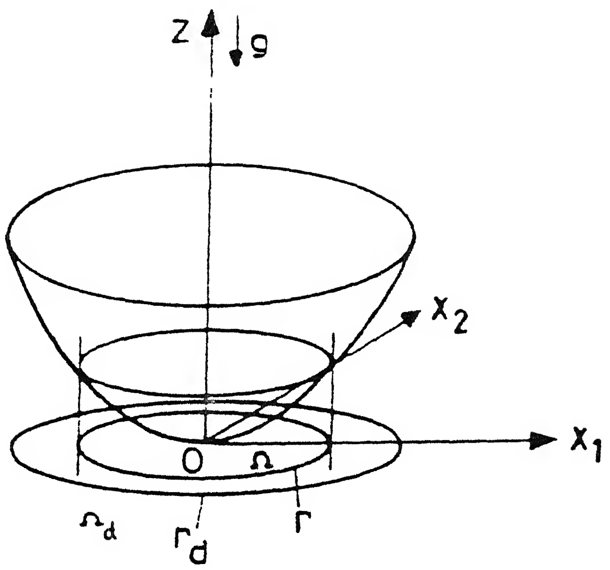


FIG.3.2 NOTATIONS

$$\int_{\Omega} (u-\psi)d\Omega = vol \qquad (3.3)$$

Expressing the potential due to gravity as

$$\bar{\pi} = gz \qquad (3.4)$$

the gravitational potential energy can be expressed as

$$U_g = \rho_l \int_{\Omega_l} \bar{\pi} \; d\Omega_l \qquad (3.5)$$

Thus substituting (3.4) in (3.5) we get

$$U_g = \frac{\rho_l g}{2}\int_{\Omega} (u^2-\psi^2)d\Omega \qquad (3.6)$$

Let $\mathcal{M}$ be the set of admissible domains $\bar{\Omega}$, which are non-empty closed and bounded subsets of $\mathbb{R}^2$ defined as follows :

$\mathcal{M} = \{\bar{\Omega} : \bar{\Omega}_a \subset \bar{\Omega} \subset \bar{\Omega}_d,\; \Omega_a, \Omega, \Omega_d$ satisfy the cone-condition [Theorem 2.1 (Chapter II)]. The metric over is defined through the Hausdorff metric $d_H(.,.)$. For any $\bar{\Omega} \in \mathcal{M}$ let V_Ω define the following :

$V_\Omega = \{u \in H^1(\Omega) : \|u\|_1 \le C$ (C is large and independent of Ω); $u \ge \psi \;\forall\; \underset{\sim}{x} \in \Omega$ and $\int_{\Omega} (u-\psi)\; d\Omega = vol\}$.

Let u_Ω be the unique solution in V_Ω of a given problem over Ω and let $G = \{(\Omega, u_\Omega) : \bar{\Omega} \in \mathcal{M}\}$.

Let us denote $(\rho g/\sigma_l)$ by bo.

Using equation (3.2), (3.3), (3.4), (3.5), (3.6) the energy expression (3.1) takes the following form :

$$U(\Omega,u) = \int_{\Omega} [1+|\underset{\sim}{\nabla} u|^2]^{1/2}dx + \frac{b\sigma}{2}\int_{\Omega}(u^2-\psi^2)\ dx-\cos\ \theta_c \int_{\Omega} [1+|\underset{\sim}{\nabla} \psi|^2]^{1/2}dx$$

$$+ \left(\frac{\sigma_s}{\sigma_l}\right)\int_{\Omega_d} [1+|\underset{\sim}{\nabla} \psi|^2]^{1/2}\ dx\ \forall\ u \in V_{\Omega} \text{ and } \bar{\Omega} \in \mathcal{M} \tag{3.7}$$

Let us denote $U(\Omega,u_{\Omega})$ as $J(\Omega)$.

3.4 PROBLEM DEFINITION AND MATHEMATICAL DETAILS :

The equilibrium-surface problem can be treated as "optimal-shape design" problem as follows :

Determine $\bar{\Omega}^* \in \mathcal{M}$ such that

$$J(\Omega^*) \le J(\Omega)\ \forall\ \bar{\Omega} \in \mathcal{M} \tag{P}$$

Then (Ω^*, u_{Ω^*}) is called an optimal pair. For a general reference on the optimal shape design problems one may refer to Pironneau [1984] and Hässlinger [1988].

We now proceed to prove the existence of a solution of (P) with the help of the following theorems and lemmas.

Theorem 3.1 : Let $\{\bar{\Omega}_m\}$ *be a sequence of a closed and bounded sets in* $\mathcal{M}$. *Then* $\exists$ a *subsequence* $(\bar{\Omega}_{mj})$ *which converges to a compact set* $\bar{\Omega} \in \mathcal{M}$ *in the Hausdorff sense.*

Proof : See [Pironneau (1984), p. 31 Theorem 1].

Lemma 3.1 : Let $I(u) = \int_{\Omega} (1+|\underset{\sim}{\nabla} u|^2)^{1/2} + \frac{b\sigma}{2}\int_{\Omega} u^2\ \forall\ u\in H^1(\Omega)$.

Let $u_n \xrightarrow{s} u$ in $H^1(\Omega)$

then $\lim_{n\to\infty} I(u_n) = I(u)$.

Proof : Let $\underset{\sim}{v}_n = \nabla \underset{\sim}{u}_n$

It is clear that $v_n \xrightarrow{s} \tilde{v}$ in $L_2(\Omega)$.

Let $I_1(\underset{\sim}{v}_n) = \int_\Omega (1+|\underset{\sim}{v}_n|^2)^{1/2}$ and $I_2(u_n) = \frac{bo}{2}\int_\Omega u_n^2$.

We have $\lim\limits_{n\to\infty} I_2(u_n) = I_2(u)$

$$|I_1(\underset{\sim}{v}_n)-I(\underset{\sim}{\tilde{v}})| = \left| \int_\Omega \frac{(1+|\underset{\sim}{v}_n|^2) - (1+|\underset{\sim}{\tilde{v}}_n|^2)}{(1+|\underset{\sim}{v}_n|^2)^{1/2} + (1+|\underset{\sim}{\tilde{v}}_n|^2)^{1/2}} \right|$$

$$\le \int_\Omega \left| |\underset{\sim}{v}_n|^2 - |\underset{\sim}{\tilde{v}}|^2 \right|$$

$\Longrightarrow \lim\limits_{n\to\infty} I_1(\underset{\sim}{v}_n) = I_1(\underset{\sim}{v})$

Thus the lemma is proved.

Now we make some remarks which will be useful subsequently.

Let $\Omega \subset\subset \tilde{\Omega}$ ($\tilde{\Omega}$ satisfies the cone-condition).

$\exists$ a linear map $P_\Omega : H^1(\Omega) \longrightarrow H^1(\tilde{\Omega})$: $P_\Omega u = u$ ($u \in H^1(\tilde{\Omega})$ s.t. $P_\Omega\big|_\Omega = u$ and also $\|P_\Omega\| \le K$ (K is a constant independent of Ω).

The second assertion is due to Chenais [1975].

Lemma 3.2 : Let $\Omega_n, \Omega \subset\subset \tilde{\Omega}$ and $v_n, v \in H^1(\tilde{\Omega})$ s.t. $v_n\big|_{\Omega_n} \in V_{\Omega_n}$

Let $v_n \xrightarrow{w} v$ in $H^1(\tilde{\Omega})$ *as* $\Omega_n \xrightarrow{d_H} \Omega$.

Then $v\big|_\Omega \in V_\Omega$.

Proof : let us define the following

(i) $\quad w = v_n - \psi$

(ii) $\quad w_n^- = (|v_n - \psi| - (v_n - \psi))/2$

(iii) $$j_1(w_n) = \frac{1}{2}\left(\int_{\Omega_n} w_n - vol\right)^2$$

(iv) $$j_2(w_n^-) = \frac{1}{2}\int_{\Omega_n} (w_n^-)^2.$$

Let D_m be a region defined as follows :

$\bar{D}_m \in M$ and $D_m \subset\subset \Omega$ s.t. $D_m \subset \Omega_n \ \forall\ n > n_m$

and $D_m \xrightarrow{d_H} \Omega$ as $m \to \infty$.

Clearly $\Omega_n = D_m \cup C_m$ where $C_m = \Omega_n \backslash D_m$.

From Rellich's Compactness Theorem we get

$$v_n \xrightarrow{s} v \text{ in } L_2\ (\tilde{\Omega})$$

Thus $w_n \xrightarrow{s} w$ in $L_2\ (\tilde{\Omega})$

$$\begin{aligned} |w_n^- - w^-| &= \frac{1}{2}\,|(v_n-\psi) - (v-\psi) + (v-v_n)| \\ &\le \frac{1}{2}\,(|v-v_n| + |v-v_n|) \\ &\le |v-v_n| \end{aligned}$$

$\Rightarrow \quad w_n \xrightarrow{s} w^-$ in $L_2\ (\tilde{\Omega})$.

$$\begin{aligned} j_1(w_n) &= \frac{1}{2}\left(\int_{\Omega_n} w_n - vol\right)^2 \\ &= \frac{1}{2}\left(\int_{D_m} w_n - vol - \int_{C_m} w_n\right)^2 \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} (j_1(w_n)) = \frac{1}{2}\left(\int_{D_m} w - vol - \int_{C_m} w_n\right)^2$$

$$\Rightarrow \lim_{m\to\infty}\ \lim_{n\to\infty} (j_1(w_n)) = \frac{1}{2}\left(\int_{\Omega} w - vol\right)^2 = 0$$

$$\begin{aligned} j_2(w_n^-) &= \frac{1}{2}\left(\int_{\Omega_n} (w_n^-)^2\right. \\ &= \frac{1}{2}\int_{D_m} (w_n^-)^2 + \int_{C_m} (\,.\,) \end{aligned}$$

$$\Rightarrow \lim_{m\to\infty} \left(\lim_{n\to\infty} j_2(w_n) \right) = \frac{1}{2} \int_{\Omega_o} (w^-)^2 = 0$$

$\Longrightarrow v|_\Omega \in V_\Omega.$

Lemma 3.3 : The graph G corresponding to (P) is sequentially compact in the following sense :

Let $\{\Omega_n\}$ $(\bar{\Omega}_n \in M)$ *be a sequence. Then* $\exists$ *a subsequence*

$$\left\{\left[\Omega_{n_k}, u_{\Omega_{n_k}}\right]\right\} \subset \left\{\left[\Omega_n , u_{\Omega_n}\right]\right\} \text{ and an element } (\Omega, u_\Omega) \in G \text{ s.t.}$$

$$\left.\begin{array}{l} \Omega_{n_k} \xrightarrow{d_H} \Omega \\ \\ u_{\Omega_{n_k}} \xrightarrow{w} \tilde{u}_\Omega \end{array}\right\} \text{ as } k \longrightarrow \infty$$

Proof : First of all let us note the graph G is non-empty since

$$\tilde{U}_{\Omega_n} = \min_{v \in V_{\Omega_n}} \int_{\Omega_n} (1+|\underset{\sim}{\nabla} u|^2)^{1/2} + \frac{bo}{2} \int_{\Omega_n} v^2 \qquad (P_1)$$

exists (see Baiocchi [1984], p.13) in view of the fact that V_{Ω_n} is convex, closed and bounded subset of $H^1(\Omega_n)$. Let the unique solution of (P_1) be u_{Ω_n} denoted by u_n (for convenience). From the definition of V_{Ω_n} and the uniform extension property we get $\|u_n\|_{1,\Omega_n} \le K$. Thus $\exists$ a weakly convergent subsequence $\{u_{n_k}\}$ which converges to $\tilde{u} \in H^1(\tilde{\Omega})$. Let us define $u_\Omega = \tilde{u}|_\Omega$. Now we shall prove that u_Ω is a solution of

(P_1) with Ω_n replaced by Ω.

let $v_{n_j} \longrightarrow v$ in $H^1(\tilde{\Omega})$ with $v_{n_j}\big|_{\Omega_{n_j}} \in V_{\Omega_{n_j}}$

with the notations used in the lemma 2 we get from the minimality of $\tilde{U}_{\Omega_n}$ in (P_1)

$$\int_{\Omega_{n_j}} \left[\left(1 + |\nabla \underset{\sim}{v}_{n_j}|^2\right)^{1/2} - \left(1 + |\nabla \underset{\sim}{v}_{n_j}|^2\right)^{1/2}\right] + \frac{bo}{2} \int_{\Omega_{n_j}} \left(v_{n_j}^2 - u_{n_j}^2\right) \geq 0$$

$$\Rightarrow \int_{D_m} \left[\left(1 + |\nabla \underset{\sim}{v}_{n_j}|^2\right)^{1/2} - \left(1 + |\nabla \underset{\sim}{u}_{n_j}|^2\right)^{1/2}\right] + \frac{bo}{2} \int_{D_m} \left(v_{n_j}^2 - u_{n_j}^2\right) + \int_{C_m} (\ ) \geq 0$$

From the lower-semicontinuity and the continuity property (Lemma 1) of the functional we get as $n_j \longrightarrow \infty$

$$\int_{D_m} \left[\left(1 + |\nabla \underset{\sim}{u}|^2\right)^{1/2} - \left(1 + |\nabla \underset{\sim}{u}_{\Omega}|^2\right)^{1/2}\right] + \frac{bo}{2} \int_{D_m} \left(v^2 - u_{\Omega}^2\right) + \int_{C_m} (.) \geq 0$$

Passing to the limit as $m \to \infty$ we get

$$\int_{\Omega} \left[\left(1 + |\nabla \underset{\sim}{v}_{n_j}|^2\right)^{1/2} - \left(1 + |\nabla \underset{\sim}{u}_{\Omega}|^2\right)^{1/2}\right] + \frac{bo}{2} \int_{\Omega} \left(v^2 - u_{\Omega}^2\right) \geq 0$$

Also as $u_\Omega \in V_\Omega$ [from lemma 3.2] u_Ω is a solution of (P_1).

Lemma 3.4 : Let $\Omega_n \xrightarrow{d_H}$ and u_n $\tilde{u}$ in $H^1(\Omega)$ where $v_n\big|_{\Omega_n} \in V_{\Omega_n}$ and $\tilde{u}\big|_{\Omega} \in V_\Omega$.

Then $\qquad U(\Omega, \tilde{u}) \leq \liminf_{n\to\infty} U(\Omega_n, u_n)$

Proof : With the help of the notations used in the previous lemma we have

$$U(\Omega_n, u_n) = \int_{D_m}\left[\left(1 + |\underset{\sim}{\nabla} u_n|^2\right)^{1/2} + \frac{b\sigma}{2}\int_{D_m} u_n^2 - \frac{b\sigma}{2}\int_{D_m}\psi^2 - \cos\theta_c \int_{D_m}\left(1 + |\underset{\sim}{\nabla}\psi_n|^2\right)^{1/2}\right.$$

$$+\left(\frac{\rho g}{\sigma_l}\right)\int_{\Omega_d}\left(1 + |\underset{\sim}{\nabla}\psi_n|^2\right)^{1/2} + \int_{C_m}(.)$$

$\because$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} U(\Omega_n, v_n)\right) \geq \lim_{m\to\infty}\left\{\int_{D_m}\left(1 + |\underset{\sim}{\nabla}\tilde{u}|^2\right)^{1/2} + \frac{b\sigma}{2}\int_{D_m}(\tilde{u}-\psi^2)\right.$$

$$\left.-\cos\theta_c \int_{D_m}\left(1 + |\underset{\sim}{\nabla}\psi_n|^2\right)^{1/2} + \left(\frac{\rho g}{\sigma_l}\right)\int_{\Omega_d}\left(1 + |\underset{\sim}{\nabla}\psi|^2\right)^{1/2} + \int_{C_m}(.)\right\}$$

$$\Longrightarrow \lim_{n\to\infty} U(\Omega_n, v_n) \geq U(\Omega, \tilde{u}).$$

Theorem 3.2 : *Let the lemmas 3.3 and 3.4 be satisfied. Then* $\exists$ *atleast one* $\Omega^*, [\bar{\Omega}^* \in \mathcal{M}]$ *which is a solution of (P).*

Proof : See [Hässlinger [1988], p. 29, Theorem 2.1].

3.5 THE DISCRETIZED PROBLEM :

Let Ω be discretized to Ω_h by triangulation T_h. T_h is composed of 3-noded linear triangles [Ciarlet [1978]] such that the vertices situated on Γ_h also belong to Γ. We also have $\mathcal{M}_h = \{\bar{\Omega}_h : \bar{\Omega}_{a_h} \subset \bar{\Omega}_h \subset \bar{\Omega}_{d_h}\}$. In such a triangulation we consider the finite element spaces. $V^h_{\Omega_o}$, V^h_{Ω} and W^h_{Ω} defined as follows :

$$V^h_{\Omega} = \{v_h \in H^1(\Omega_h) \cap C^o(\bar{\Omega}_h) : v_h > \psi \text{ in } \Omega_h : \int_{\Omega_h}(v_h - \psi)\, dx = vol : \|v_h\|_1 \leq C$$

$$v_h|_{T_j} \in P_1(T_j) \text{ and } v_h|_{\Gamma_h} = \psi_{\Gamma_h} \text{ at the nodes}\}$$

$$V^h_{\Omega_O} = \{v_h \in H^1_O(\Omega_h) \cap C^O(\bar{\Omega}_h) : v_h|_{T_j} \in P_1(T_j)\}$$

$$W^h_{\Omega_O} = \{w_h \in H^1_O(\Omega_h) \cap C^O(\bar{\Omega}_h) : w_h|_{T_j} \in P_1(T_j) : w_h|_{T_j} = \psi_{\Gamma_h} \text{ at the nodes}\}$$

(where $T_j \in T_h$ and $P_1(T_j)$ is the space of linear polynomials on T_j).

$$\text{Let } E_h = \left(\inf_{\forall u_h \in V^h_\Omega} \left\{ \int_{\Omega_h} \left[1 + |\underset{\sim}{\nabla} v_h|^2\right]^{1/2} dx + \int_{\Omega_h} \frac{bo}{2} (v_h^2 - \psi^2) dx \right\} \right)$$

and

$$A_{oh} = \frac{\sigma_g}{\sigma_l} \int_{\Omega_{d_h}} \left[1 + |\underset{\sim}{\nabla}\psi_h|^2\right]^{1/2} dx \; : \; E_{h1} = -\cos\theta_c \int_{\Omega_h} \left[1 + |\underset{\sim}{\nabla}\psi_h|^2\right]^{1/2} dx$$

$J_h(\Omega_h) = E_h + E_{h1} + A_{oh}$.

Now the discrete problem is as follows :

$$\text{Determine } \Omega_h^* \text{ such that } J_h(\Omega_h^*) \le J_h(\Omega_h) \forall \; \Omega_h \in \mathcal{M}_h \qquad (P_h)$$

In order to determine E_h we solve the weak problem which is given below :

Determine $u_h \in W^h_\Omega$ such that

$$a_h(u_h, v_h) + \frac{1}{\epsilon_A} (\beta_{1h}(u_h, v_h)) + \frac{1}{\epsilon_B} (\beta_{2h}(u_h, v_h)) = 0 \qquad (P_{h1})$$

$$\forall v_h \in V^h_{\Omega o}$$

$$\text{where } a_h(u_h, v_h) = \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h \cdot \underset{\sim}{\nabla} v_h}{[1 + |\underset{\sim}{\nabla} u_h|^2]^{1/2}} dx + bo \int_{\Omega_h} u_h v_h \, dx$$

$$\beta_{1h}(u_h, v_h) = \left(\int_{\Omega_h} f\ (u_h - \psi)\ dx - vol\right)\left(\int_{\Omega_h} v_h dx\right)$$

$$\beta_{2h}(u_h, v_h) = \int_{\Omega_h} M(u_h - v_h) v_h\ dx$$

and $f = 1$ for $u_h > \psi$; and $f = 0$ otherwise $M = 0$ for $u_h > \psi$ and $M = 1$ otherwise $\epsilon_A, \epsilon_B > 0$ are penalty parameters.

In the ensuing numerical scheme for the solution of (P_h) we require the gradient of $J_h(\Omega_h)$. For this we develop the necessary tools.

It can be seen that (P_h) can be cast into a variational inequality formulation. [Duvaut and Lions [1976], Glowinski [1981]].

Lemma 3.4 : Let T_j *be the element obtained from* $T_j \in J_h$ *by translating one of its vertices* $\underset{\sim}{q}^k$ *to* $\underset{\sim}{q}'^k$ $(\underset{\sim}{q}'^k = \underset{\sim}{q}^k + \delta\underset{\sim}{q}^k)$ *(Fig. 3.3 Let* $N^i(.)$ *be the basis function associated with the vertex* $\underset{\sim}{q}^i$.

Then $N^i(\underset{\sim}{x}) - N'^i(\underset{\sim}{x}) = -N^k(\underset{\sim}{x}).\nabla N^i(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k + O(|\delta\underset{\sim}{q}^k|)\ \forall\ x \in T_j \cap T_j^1$

$\forall\ \underset{\sim}{q}^k, \underset{\sim}{q}^i$ *vertices of* T_j

(3.8)

Proof : See [Pironneau [1984], *pg. 102*].

Lemma 3.5 : Let g *be a continuously differentiable function on* T_j*. Let* T_j' *be obtained from* T_j *by translating* $\underset{\sim}{q}^k$ *to* $\underset{\sim}{q}^k + \delta\underset{\sim}{q}^k$*. Then we have*

$$\int_{T_j' - (T_j \cap T_j')} g\ dx - \int_{T_j' - (T_j \cap T_j')} g.dx = \int_{T_j} \delta\underset{\sim}{q}^k.\ \nabla(gN^k) + O(|\delta\underset{\sim}{q}^k|) \quad (3.9)$$

Proof : See [Pironneau [1984], *pg. 102*].

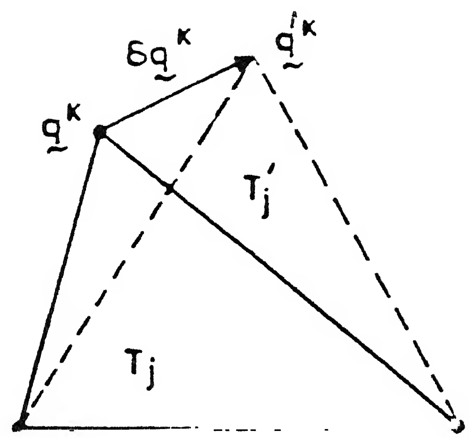


FIG. 3.3 DEFORMATION OF T_j BY SHIFTING OF q^{K}

Let us consider the variation of the solution when the node $\underset{\sim}{q}^k$ of $T_j \in T_h$ *is shifted to* $\underset{\sim}{q}^k + \delta\underset{\sim}{q}^k$ with the help of the following lemmas.

Lemma 3.6 : Let K denote the set of boundary nodes of Ω_h and K^c the set of interior nodes of Ω_h. Let the node $\underset{\sim}{q}^k$ of T_j be shifted to $\underset{\sim}{q}^k + \delta\underset{\sim}{q}^k$. Then we have the following :

$$\delta u_h = \delta\tilde{u}_h + N^k X_h \underset{\sim}{\nabla}\psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k - N^k \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\ \delta\underset{\sim}{q}^k + O(|\delta\underset{\sim}{q}^k|) \qquad (3.10)$$

where X_h is a characteristic function defined as follows

$$X_h = 0 \text{ for } \underset{\sim}{q}^k \in \Gamma_h$$
$$= 1 \text{ for } \underset{\sim}{q}^k \in \Gamma_h$$

Proof : We have $u_h = \sum_{i\in K^c} u_i N^i + \sum_{i \in K} u_i N^i$.

$$\therefore \delta u_h = \sum_{i\in K} (\delta u_i N^i + u_i\ \delta N^i) + \sum_{i\in K} (\delta u_i N^i + u_i\ \delta N^i) \qquad (3.11)$$

From (3.8) we get $\delta N^i = -N^k \underset{\sim}{\nabla} N^i(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k$ which we substitute in (3.11) to get

$$\delta u_h = \sum_{i\in K} \delta u_i N^i - \sum_{i\in K} N^k u_i \underset{\sim}{\nabla} N^i(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k + \sum_{i\in K} \delta_{ik} X_h \underset{\sim}{\nabla}\ \psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k N^i$$
$$- \sum_{i\in K} N^k u_i \underset{\sim}{\nabla} N^i(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k \qquad (3.12)$$

$$\text{But } \underset{\sim}{\nabla} u_h = \sum_{i\in K} \underset{\sim}{\nabla}(u_i N^i) + \sum_{i\in K^c} \underset{\sim}{\nabla}(u_i N^i) = \sum_{i\in K} u_i\ \underset{\sim}{\nabla} N^i + \sum_{i\in K^c} u_i\ \underset{\sim}{\nabla} N^i \qquad (3.13)$$

Substituting (3.13) in (3.12) we get

$$\delta u_h = \sum_{i\in K} \delta u_i N^i + N^k X_h \underset{\sim}{\nabla}\ \psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k - N^k \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\ \delta\underset{\sim}{q}^k.$$

We define $\delta \tilde{u}_h^{*} = \sum_{i \in \mathcal{K}} \delta u_i N^i = \sum_{i \in \mathcal{K}} \frac{\partial u_i}{\partial \underset{\sim}{q}^k} \cdot \delta \underset{\sim}{q}^k N^i$.

Thus we get (3.10). It is readily seen that $\delta \tilde{u}_h^{*} \in V_{\Omega_0}^h$.

Lemma 3.7 : Let the node $\underset{\sim}{q}^k$ *of* T_j *be translated to* $\underset{\sim}{q}^k + \delta \underset{\sim}{q}^k$, *then the following hold* $\forall\ u_h \in W_\Omega^h$ *and* $\forall u_h, \delta \tilde{v}_h^{*} \in V_{\Omega_o}^h$

$$(i)\ \delta a_h(u_h, v_h) = \int_{\Omega_h} \frac{\underset{\sim}{\nabla} (\delta \tilde{u}_h^{*}) . \underset{\sim}{\nabla} u_h}{\sqrt{g_h}} dx - \int_{\Omega_h} \frac{\underset{\sim}{\nabla} \tilde{u}_h . \underset{\sim}{\nabla} (\delta \tilde{u}_h^{*})}{g_h^{3/2}} (\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} v_h)\ dx$$

$$+ \int_{\Omega_h} bO\ \delta \tilde{u}_h^{*}\ v_h dx$$

$$+ \int_{\Omega_h} \frac{1}{\sqrt{g_h}} (\underset{\sim}{\nabla} (N^k X_h \underset{\sim}{\nabla} \psi_{\Gamma_h} (\underset{\sim}{q}^k) . \underset{\sim}{\nabla} v_h - \underset{\sim}{\nabla} (N^k \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k) . \underset{\sim}{\nabla} v_h) dx$$

$$+ \int_{\Omega_h} \frac{1}{g_h^{3/2}} \Big[(\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} (N^k \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k)) (\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} v_h)$$

$$- (\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} (N^k \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k) (\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} v_h) \Big] dx$$

$$- \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h}{\sqrt{g_h}} . \underset{\sim}{\nabla} (N^k\ \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) .\ \delta \underset{\sim}{q}^k) dx + bo \int_{\Omega_h} N^k (\underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) .\ \delta \underset{\sim}{q}^k$$

$$- \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) .\ \delta \underset{\sim}{q}^k) v_h\ dx - bo \int_{\Omega_h} u_h\ N^k\ \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k) .\ \delta \underset{\sim}{q}^k\ dx$$

$$+ \int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla} \Big[\Big[\frac{(\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} v_h)}{\sqrt{g_h}} \Big] N^k \Big] dx + bo \int_{\Omega_h} \delta \underset{\sim}{q}^k . (\underset{\sim}{\nabla} (u_h v_h N^k)\ dx$$

$$+ O(|\delta \underset{\sim}{q}^k|) \text{ where } g_h = 1 + |\underset{\sim}{\nabla} u_h|^2 \qquad (3.14)$$

(ii) $$\delta(\beta_{1h}(u_h),v_h) = (\int_{\Omega_h} \delta \tilde{u}_h^* f\, dx)(\int_{\Omega_h} v_h f\, dx)$$

$$+(\int_{\Omega_h} (N^k X_h \underset{\sim}{\nabla} \psi_{\Gamma_h} (\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k - N^k \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\ \delta \underset{\sim}{q}^k) f dx)\ (\int_{\Omega_h} v_h f\, dx)$$

$$-(\int_{\Omega_h} (u_h-\psi) f\, dx - vol)(\int_{\Omega_h} N^k f \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k dx)$$

$$+ (\int_{\Omega_h} (u_h-\psi) f\, dx - vol)(\int_{\Omega_h} \delta \underset{\sim}{q}^k.\underset{\sim}{\nabla}(N^k v_h f) dx)$$

$$+(\int_{\Omega_h} \delta \underset{\sim}{q}^k.\underset{\sim}{\nabla} (f(u_h-\psi)N^k) dx)(\int_{\Omega_h} f\, v_h dx) + O(|\delta \underset{\sim}{q}^k|) \tag{3.15}$$

(iii) $$\delta(\beta_{2h}(u_h),v_h) = \int_{\Omega_h} M\, \delta \tilde{u}_h^*\, v_h dx + \int_{\Omega_h} M(N^k X_h \underset{\sim}{\nabla} \psi_{\Gamma_h} (\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k$$

$$-N^k \underset{\sim}{\nabla} u_h (\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k) v_h\, dx$$

$$- \int_{\Omega_h} M\ (u_h-\psi)\ N^k \underset{\sim}{\nabla} v_h (\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k\, dx$$

$$+ \int_{\Omega_h} \delta \underset{\sim}{q}^k.\underset{\sim}{\nabla}\ (M(u_h-\psi)v_h N^k) dx + O(|\delta \underset{\sim}{q}^k|) \tag{3.16}$$

Proof : (i) $$a_h(u_h,v_h)|_{T_j} = \int_{T_j} \frac{\underset{\sim}{\nabla} u_h\ .\underset{\sim}{\nabla} v_h}{\sqrt{1+|\underset{\sim}{\nabla} u_h|^2}} dx + bo \int_{T_j} u_h v_h\, dx$$

$$a_h'(u_h',v_h')|_{T_j} = \int_{T_j} \frac{\underset{\sim}{\nabla}\ (u_h+\delta u_h).\ \underset{\sim}{\nabla}\ (v_h+\delta u_h)}{(1+|\underset{\sim}{\nabla} u_h + \underset{\sim}{\nabla} \delta u_h|^2)^{1/2}} dx + \int_{T_j} bo(u_h+\delta u_h)(v_h+\delta v_h) dx$$

$$+\int_{T_j} \delta \underset{\sim}{q}^k.\underset{\sim}{\nabla} \left[N^k \frac{\underset{\sim}{\nabla} u_h .\ \underset{\sim}{\nabla} v_h}{g_h^{1/2}} \right] dx + \int_{T_j} bo\ \delta \underset{\sim}{q}^k.\underset{\sim}{\nabla}(u_h v_h N^k) dx + O(|\delta \underset{\sim}{q}^k|).$$

The last two integrals are obtained from (3.9)

$$\delta a_h(u_h,v_h)|_{T_j} = a'(u_h',v_h')|_{T_j} - a_h(u_h,v_h)|_{T_j}$$

$$= \int_{T_j} \frac{(\nabla u_h + \nabla v_h).(\nabla v_h + \nabla \delta v_h)}{g_h^{1/2}} \left[1+2 \frac{\nabla u_h . \nabla \delta v_h}{g_h} \right]^{-1/2} dx$$

$$+ bo \int_{T_j} u_h v_h \, dx + bo \int_{T_j} u_h \delta v_h dx + bo \int_{T_j} \delta u_h v_h \, dx$$

$$- \int_{T_h} \frac{(\nabla u_h . \nabla v_h)}{g_h^{1/2}} dx - bo \int_{T_j} u_h v_h dx + \int_{T_j} \delta q^k . \nabla \left(\frac{N^k \nabla u_h . \nabla v_h}{g_h^{1/2}} \right) dx$$

$$+ bo \int_{T_j} \delta q^k . \nabla (u_h v_h N^k) dx$$

Invoking (3.10) we get

$$\delta a_h(u_h,v_h)|_{T_j} = \int_{T_j} \frac{\nabla \delta \tilde{u}_h^* . \nabla v_h}{g_h^{3/2}} - \int_{T_j} \frac{1}{g_h^{1/2}} (\nabla u_h + \nabla \delta u_h^*)(\nabla u_h . \nabla v_h) dx$$

$$+ \int_{T_j} f \delta \tilde{u}_h^* v_h dx + \int_{T_j} \frac{1}{g_h^{1/2}} (\nabla (N^k X_h \nabla \psi_{\Gamma_h} (q^k) . \delta q^k$$

$$- \nabla (N^k \nabla u_h (q^k) . \delta q^k) . \nabla v_h \, dx$$

$$+ \int_{T_j} \frac{1}{g_h^{3/2}} \left[\left(\nabla u_h . \nabla (N^k \nabla u_h (q^k) . \delta q^k \right) \right]$$

$$- \left[\nabla u_h . \nabla (N^k X_h \nabla \psi_{\Gamma_h} (q^k) . \delta q^k (\nabla u_h . \nabla v_h)) \right] dx$$

$$+bo \int_{T_h} N^k x_h \nabla\psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k - N^k \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k)v_h \, dx$$

$$-bo \int_{T_j} u_h N^k \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k \, dx.$$

$$+\int_{T_h} \delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}\left[N^k\left(\frac{\underset{\sim}{\nabla u_h}.\underset{\sim}{\nabla v_h}}{g_h^{1/2}}\right)\right]dx + bo \int_{T_h} \delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(u_h v_h N^k)dx$$

But $\delta a_h(u_h,v_h) = \sum_j \delta a_h(u_h,v_h)|_{T_j}$. Hence we obtain (3.14).

(ii) $$(\beta_{1h}(u_h),v_h)|_{T_j} = (\int_{T_j}(u_h-\psi)f\,dx - vol)(\int_{T_j} v_h\, f\,dx)$$

$$(\beta_{1h}(u_h'),v_h')|_{T_j'} = (\int_{T_j'}(u_h-\psi)f\,dx - vol)(\int_{T_j'} v_h'\, f\,dx)$$

$$\delta(\beta_{1h}(u_h),v_h)|_{T_j'} = (\int_{T_j}((u_h-\psi)+\delta u_h)\, f\,dx - vol$$

$$+\int_{T_j}\delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(N^k(u_h-\psi)f)dx)(\int_{T_j}(v_h+\delta v_h)\, f\,dx$$

$$+\int_{T_j}\delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(N^k v_h f)dx) - (\int_{T_j}(u_h-\psi)f\,dx - vol)(\int_{T_j} v_h\, f\,dx)$$

$$=(\int_{T_j}\delta u_h f\,dx)(\int_{T_j} v_h\, f\,dx) - (\int_{T_j} f(u_h-\psi)dx - vol)(\int_{T_j} N^k f\,\underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k dx).$$

$$+(\int_{T_j} f(u_h-\psi)dx - vol)(\int_{T_j}\delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(v_h N^k f)dx + (\int_{T_j}\delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(fu_h-\psi)N^k)dx)(\int_{T_j} f v_h dx)$$

Invoking (3.10) and noting that $\delta(\beta_{1h}(u_h),v_h) = \sum_j \delta(\beta_{1h}(u_h),v_h)|_{T_j}$

We get (3.15).

(iii) $(\beta_{2h}(u_h),v_h)|_{T_j} = \int_{T_j} M(u_h-\psi)v_h \, dx$

$$(\beta'_{2h}(u'_h),v'_h)|_{T'_j} = \int_{T'_j} M(u'_h-\psi)v'_h \, dx = \int_{T_j} M(u'_h-\psi)v'_h \, dx$$

$$+ \int_{T_j} \delta \underset{\sim}{q}^k \, \underset{\sim}{\nabla}(M(u_h-\psi)v_h N^k)dx$$

$$= \int_{T_j} (M(u_h+\delta u_h-\psi)(v_h+\delta v_h) \, dx$$

$$+ \int_{T_j} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(M(u_h-\psi)v_h N^k)dx$$

$$\therefore \quad \delta(\beta_{2h}(u_h),v_h)|_{T_j} = \Big[(\beta'_{2h}(u'_h),v'_h) \Big]\Big|_{T'_j} - \Big[(\beta_{2h}(u_h),v_h)\Big]\Big|_{T_j}$$

$$= \int_{T_j} M \, \delta \tilde{u}^*_h \, dx$$

$$+ \int_{T_j} MN^k \Big[X_h \underset{\sim}{\nabla} \psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k - \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k \Big] v_h dx$$

$$-\int_{T_j} M(u_h-\psi)N^k \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k dx + \int_{T_j} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(M(u_h-\psi)v_h N^k) \, dx$$

As $\delta(\beta_{2h}(u_h),v_h) = \sum_j \delta(\beta_{2h}(u_h),v_h)|_{T_j}$ we get (3.16)

Theorem 3.3 : *Let* u_h, v'_h *be the solution of* P_{h1} *with trinagulations* T_h *and* T'_h *respectively, where* T'_h *is obtained from* T_h *by translating the vertex* $\underset{\sim}{q}^k$ *to* $\underset{\sim}{q}'^k$. *Let* $\delta \tilde{u}^*_h \in V^h_{\Omega o}$ *be the solution of the following problem* $\forall \; v_h \in V^h_{\Omega o}$: -

$$\int_{\Omega_h} \frac{\nabla \delta \tilde{u}_h^{*} \cdot \nabla v_h}{g_h^{1/2}}\, dx - \int_{\Omega_h} \frac{\nabla u_h \cdot \nabla \delta \tilde{u}_h^{*}}{g_h^{3/2}} (\nabla u_h \cdot \nabla v_h) dx + bo \int_{\Omega_h} \delta \tilde{u}_h^{*} v_h dx$$

$$+ \frac{1}{\epsilon_A}\left[\int_{\Omega_h} \delta \tilde{u}_h^{*} f dx\right] \left[\int_{\Omega_h} v_h f dx\right] + \frac{1}{\epsilon_A} \int_{\Omega_h} M \delta \tilde{u}_h^{*} v_h dx$$

$$= -\int_{\Omega_h} \frac{1}{g_h^{1/2}} (\nabla (N^k X_h \nabla \psi_{\Gamma_h} (q^k) . \delta q^k) - \nabla (N^k \nabla u_h (q^k) . \delta q^k)) . \nabla v_h dx$$

$$-\int_{\Omega_h} \frac{1}{g_h^{1/2}} ((\nabla u_h \; \nabla (N^k \nabla u_h (q^k) . \delta q^k)) (\nabla u_h \cdot \nabla (N^k X_h \nabla \psi_{\Gamma_h} (q^k) . \delta q^k))) (\nabla u_h \cdot \nabla v_h) dx$$

$$+ \int_{\Omega_h} \frac{\nabla u_h}{g_h^{1/2}} \cdot \nabla (N^k \; \nabla u_h \; (q^k) . \delta q^k) dx - \int_{\Omega_h} \delta q^k . \nabla \left(\left(\frac{\nabla u_h \cdot \nabla v_h}{g_h^{1/2}} \right) N^k \right) dx$$

$$-bo \int_{\Omega_h} N^k (X_h \nabla \psi_{\Gamma_h} (q^k) . \delta q^k) - \nabla u_h \; (q^k) . \delta q^k) v_h dx + bo \int_{\Omega_h} u_h N^k \nabla v_h (q^k) . \delta q^k dx$$

$$-bo \int_{\Omega_h} . \delta q^k . \; \nabla (u_h v_h N^k) dx$$

$$-\frac{1}{\epsilon_A} \left[\int_{\Omega_h} N^k (X_h \nabla \; \psi_{\Gamma_h} (q^k) . \delta q^k) - \nabla u_h \; (q^k) . \delta q^k) f dx \right] \left(\int_{\Omega_h} f v_h dx \right)$$

$$+\frac{1}{\epsilon_A} \left[\int_{\Omega_h} f (u_h - \psi) dx - vol \right] \left[\int_{\Omega_h} N^k \; f \nabla v_h \; (q^k) . \delta q^k) dx \right]$$

$$-\frac{1}{\epsilon_A} \left[\int_{\Omega_h} (u_h - \psi) f \; dx - vol \right] \left[\int_{\Omega_h} \delta q^k . \nabla \; (N^k v_h f) dx \right]$$

$$-\frac{1}{\epsilon_A}\left[\int_{\Omega_h}\delta\underset{\sim}{q}^k\cdot\ \underset{\sim}{\nabla}(f(u_h-\psi)N^k)dx)\right]\left[\int_{\Omega_h}v_h f dx\right]$$

$$-\frac{1}{\epsilon_B}\int_{\Omega_h}MN^k(X_h\underset{\sim}{\nabla}\psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k\ -\ \underset{\sim}{\nabla}u_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k)v_h dx$$

$$-\frac{1}{\epsilon_B}\int_{\Omega_h}M(u_h-\psi)N^k\underset{\sim}{\nabla}v_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k dx\ -\frac{1}{\epsilon_B}\int_{\Omega_h}\delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(M(u_h-\psi)N^k v_h)\ dx$$

(3.17)

Then $\|u_h'-u_h-\delta\tilde{u}_h^* +N^k\underset{\sim}{\nabla}\ (\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k-N^k x_h\underset{\sim}{\nabla}\ \psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k)\|_1\leq\ \bar{C}\ |\delta\underset{\sim}{q}^k|$

(3.18)

where $\tilde{C}$ is a constant which depends on Ω_h only.

Proof : Combining (3.14), (3.15) and (3.16) we get (3.17). Using (3.10), (3.17) can be written as

$$\int_{\Omega_h}\frac{\underset{\sim}{\nabla}\delta u_h\ .\underset{\sim}{\nabla}v_h}{g_h^{1/2}}\ dx\ -\ \int_{\Omega_h}\frac{\underset{\sim}{\nabla}u_h\ .\underset{\sim}{\nabla}\delta u_h}{g_h^{3/2}}\ (\underset{\sim}{\nabla}u_h.\underset{\sim}{\nabla}v_h)dx\ +\ bo\ \int_{\Omega_h}\delta u_h v_h dx$$

$$+\frac{1}{\epsilon_A}\left[\int_{\Omega_h}\delta u_h\ f\ dx\right]\left[\int_{\Omega_h}v_h\ f\ dx\right]+\frac{1}{\epsilon_B}\ \int_{\Omega_h}M\ \delta u_h v_h dx\ =$$

$$\int_{\Omega_h}\frac{\underset{\sim}{\nabla}u_h}{g_h^{1/2}}\ .\underset{\sim}{\nabla}(N^k\ \underset{\sim}{\nabla}u_h\ (\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k)dx$$

$$-\int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}\left[\left(\frac{\underset{\sim}{\nabla u_h} . \underset{\sim}{\nabla v_h}}{g_h^{1/2}}\right) N^k\right] dx + bo \int_{\Omega_h} (u_h N^k \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k) dx - bo \int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(u_h v_h N^k) d$$

$$+\frac{1}{\epsilon_A}\left[\int_{\Omega_h} f\ (u_h-\psi)\ dx - vol\right]\left[\int_{\Omega_h} N^k\ f\ \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k\ dx\right]$$

$$-\frac{1}{\epsilon_A}\left[\int_{\Omega_h} f\ (u_h-\psi)\ dx - vol\right]\left[\int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(N^k v_h f)\ dx\right]$$

$$-\frac{1}{\epsilon_A}\left[\int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(f(u_h-\psi)N^k) dx\right]\left[\int_{\Omega_h} v_h f\ dx\right]$$

$$+\frac{1}{\epsilon_B}\int_{\Omega_h} M(u_h-\psi)N^k \underset{\sim}{\nabla} v_h(\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k dx - \frac{1}{\epsilon_B}\int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(M(u_h-\psi)N^k v_h)\ dx$$

(3.19)

Let us define : $\tilde{bo} = bo + \frac{M}{\epsilon_B}$

and $$A_h(w_h, v_h) = \int_{\Omega_h} \frac{\underset{\sim}{\nabla w_h} . \underset{\sim}{\nabla v_h}}{g_h^{1/2}}\ dx - \int_{\Omega_h} \frac{(\underset{\sim}{\nabla w_h} . \underset{\sim}{\nabla v_h})(\underset{\sim}{\nabla w_h} . \underset{\sim}{\nabla v_h})}{g_h^{3/2}}\ dx$$

$$+\int_{\Omega_h} bo\ w_h v_h\ dx + \frac{1}{\epsilon_A}\left[\int_{\Omega_h} w_h\ f\ dx\right]\left[\int_{\Omega_h} v_h\ f\ dx\right] \qquad (3.20)$$

It is clear that $A_h(w_h, w_h) \geq 0$

$$A_h(w_h, w_h) = 0 \implies w_h = 0$$

$$(A_h(w_h, w_h))^{1/2} \leq A_h(w_h, w_h)^{1/2} + A_h(v_h, v_h)^{1/2}$$

hence $(A_h(w_h,w_h))^{1/2} = \|w_h\|_{A_h}$ qualifies as a norm.

Let us show the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_{A_h}$.

Define $\underset{\sim}{\nabla u_h} = (u_{h_1}\ u_{h_2})^T$ and $\underset{\sim}{\nabla W_h} = (w_{h_1}\ w_{h_2})^T$ for $\Omega_h \subset\subset \mathbb{R}^2$.

Substituting in (3.20) we get

$$A_h(w_h,w_h) = \int_{\Omega_h} \frac{1}{g_h^{3/2}} \left\{(1+u_{h_1}^2+u_{h_2}^2)(w_{h_1}^2+w_{h_2}^2)-(u_{h_1}w_{h_1}+u_{h_2}w_{h_2})^2\right\}dx$$

$$+\int_{\Omega_h} \tilde{b}o\, w_h^2\, dx + \frac{1}{\epsilon_A}\left[\int_{\Omega_h} w_h\, f\, dx\right]\left(\int_{\Omega_h} w_h\, f\, dx\right)$$

$$= \int_{\Omega_h} \frac{1}{g_h^{3/2}}\left[\left\{(1+u_{h_1}^2\right) w_{h_1}^2 + (1+u_{h_2}^2) w_{h_2}^2 - 2u_{h_1}u_{h_2}w_{h_1}w_{h_2}\right] dx + \int_{\Omega_h} \tilde{b}o\, w_h^2\, dx$$

$$-\frac{1}{\epsilon_A}\left(\int_{\Omega_h} w_h\, f\, dx\right)^2 \le C_1\,|w_{h_1}|_1^2 + C_2|w_h|_o^2 \le C_3^2\,\|w_h\|_1^2$$

Also we find the $g_h \le 1 + C^2$ (as $\|u_h\|_1 \le C$).

$$\therefore\ A_h(w_h,v_h) \ge \int_{\Omega_h} \frac{\underset{\sim}{\nabla w_h}\cdot\underset{\sim}{\nabla w_h}}{g_h^{1/2}}\, dx + \int_{\Omega_h} \tilde{b}o\, w_h^2\, dx \ge C_4^2\, |w_h|_1^2\ .$$

From the equivalence of semi-norms we get

$$C_3\,\|w_h\|_1 \ge \|w_h\|_{A_h} \ge C_4\,\|w_h\|_1.$$

The R.H.S. of (19) $\leq C_o|\delta \underset{\sim}{q}^k|$

$$\Longrightarrow \|\delta u_h\|_1 \leq \tilde{C}|\delta \underset{\sim}{q}^k|$$

$$\Longrightarrow \|u_h' - \delta\tilde{u}_h^* - u_h + N^k \nabla v_h(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k - N^k X_h \nabla\psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k\| \leq \tilde{C}\;|\delta \underset{\sim}{q}^k|.$$

Theorem 3.3 shows that a small change in the node induces a small change in the solution.

Corollary 3.1 : *Let* $\delta\tilde{u}_h^* \in V^h_{\Omega o}$ *be a solution of* (3.17). *Let*

$\underset{\sim}{\theta}_h^k = \Sigma \dfrac{\partial u_i}{\partial q^k} N^i$ *and* θ^k_{hl} ($l = 1,2.....n$) *denote the components of* $\underset{\sim}{\theta}_h^k$

(where n is the dimension of $\Omega_h \subset\subset \mathbb{R}^n$). *Then we have the following:* $-\ \forall\ v_h \in V^h_{\Omega o}$:$-$

$$\int_{\Omega_h} \frac{\underset{\sim}{\nabla}(\theta^k_{hl}).\underset{\sim}{\nabla} v_h}{g_h^{1/2}}\,dx - \int_{\Omega_h} \frac{1}{g_h^{3/2}}(\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(\theta^k_{hl})).(\nabla u_h.\underset{\sim}{\nabla} v_h)dx + \int_{\Omega_h} bo\,(\theta^k_{hl}).v_h$$

$$+\frac{1}{\epsilon_A}\left[\int_{\Omega_h} f\,\theta^k_{hl}\,dx\right]\left[\int_{\Omega_h} v_h\,f\,dx\right] + \frac{1}{\epsilon_B}\int_{\Omega_h} M\,\theta^k_{hl} v_h\,dx$$

$$= -\int_{\Omega_h} \frac{1}{g_h^{1/2}}(\underset{\sim}{\nabla} N^k X_h \frac{\partial\psi}{\partial X_l}(\underset{\sim}{q}^k) - \underset{\sim}{\nabla}(N^k \frac{\partial u_h}{\partial X_l}(q^k))).\underset{\sim}{\nabla} v_h\,dx$$

$$-\int_{\Omega_h} \frac{1}{g_h^{3/2}}((\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(N^k \frac{\partial u_h}{\partial X_l}(q^k))-(\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(N^k X_h \frac{\partial\psi}{\partial x_l}))))(\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla} u_h)\,dx$$

$$- \int_{\Omega_h} \frac{1}{g_h^{1/2}} \, . \, \underset{\sim}{\nabla} \left(N^k \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k) \right) dx - \int_{\Omega_h} \frac{\partial}{\partial x_l} \left[\left(\frac{\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} v_h}{g_h^{1/2}} \right) N^k \right] dx$$

$$-bo \int_{\Omega_h} N^k \left(X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k) \right) v_h \, dx + bo \int_{\Omega_h} u_h N^k \, \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k) \, dx$$

$$-bo \int_{\Omega_h} \frac{\partial}{\partial x_l} (u_h v_h N^k) \, dx - \frac{1}{\epsilon_A} \left(\int_{\Omega_h} N^k \left(X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l} \right) f \, dx \right) \left(\int_{\Omega_h} f v_h \, dx \right)$$

$$+ \frac{1}{\epsilon_A} \left(\int_{\Omega_h} f(u_h - \psi) dx - vol \right) \left(\int_{\Omega_h} f \, N^k \, \frac{\partial v_h}{\partial x_l} (\underset{\sim}{q}^k) \, dx \right)$$

$$- \frac{1}{\epsilon_A} \left(\int_{\Omega_h} f(u_h - \psi) dx - vol \right) \left(\int_{\Omega_h} \frac{\partial}{\partial x_l} (N^k v_h \, f) \, dx \right)$$

$$- \frac{1}{\epsilon_A} \left(\int_{\Omega_h} \frac{\partial}{\partial x_l} (f \, (u_h - \psi) \, N^k) \, dx \right) \left(\int_{\Omega_h} v_h \, f \, dx \right)$$

$$- \frac{1}{\epsilon_B} \left(\int_{\Omega_h} M N^k \left(X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k) - \frac{\partial v_h}{\partial x_l} (\underset{\sim}{q}^k) \right) v_h dx \right. + \frac{1}{\epsilon_B} \int_{\Omega} M(u_h - \psi) N^k \, \frac{\partial v_h}{\partial x_l} (\underset{\sim}{q}^k) \, dx$$

$$- \frac{1}{\epsilon_B} \int_{\Omega_h} \frac{\partial}{\partial x_l} (M(u_h - \psi) \, v_h N^k) \, dx \qquad l = 1, 2, \ldots, n \qquad (3.21)$$

Proof : This is a straight forward application of

$\delta \tilde{u}_h^* = \Sigma \, \delta u_i N^i = \Sigma \, \dfrac{\partial u_i}{\partial \underset{\sim}{q}^k} N^i . \delta \underset{\sim}{q}^k$ into (3.17).

Theorem 3.4 : *Let $p_h \in V^h_{\Omega o}$ be a solution of the following problem*

$$\int_{\Omega_h}\left\{\frac{1}{g_h^{1/2}}\left((\nabla \underset{\sim}{p}_h . \nabla \underset{\sim}{v}_h - \frac{1}{g_h^{1/2}}(\nabla \underset{\sim}{u}_h . \nabla \underset{\sim}{p}_h)(\nabla \underset{\sim}{u}_h . \nabla \underset{\sim}{v}_h)\right)dx\right.$$

$$+ \int_{\Omega_h} b\tilde{o}\ p_h v_h dx + \left[\int_{\Omega_h} p_h\, f\, dx\right]\left[\int_{\Omega_h} v_h\, f\, dx\right]$$

$$= \int_{\Omega_h} \frac{\nabla \underset{\sim}{u}_h . \nabla \underset{\sim}{v}_h}{g_h^{1/2}}\, dx + \int_{\Omega_h} bo\ u_h v_h dx \quad \forall\ v_h \in V^h_{\Omega o} \tag{3.22}$$

with $\tilde{bo} = bo + \dfrac{M}{\epsilon_B}$.

$$\text{Then } \frac{\partial J^h}{\partial \underset{\sim}{q}^k_l} = T^{k,l}_{h1} + T^{k,l}_{h2} + T^{k,l}_{h3},\quad l = 1,2,\ldots\ldots n \tag{3.23}$$

$$\text{where } T^{k,l}_{h1} = \int_{\Omega_h} \frac{1}{g_h^{1/2}}(\nabla \underset{\sim}{u}_h . \nabla (N^k X_h \frac{\partial \psi}{\partial x_l}(\underset{\sim}{q}^k) - \nabla \underset{\sim}{u}_h . \nabla (N^k \frac{\partial u_h}{\partial x_l}(\underset{\sim}{q}^k)))\, dx$$

$$+ \int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k g_h^{1/2})\, dx + \frac{bo}{2}\int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k u_h^2) dx - \frac{bo}{2}\int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k \psi^2)\, dx$$

$$+ bo \int_{\Omega_h}(u_h N^k X_h \frac{\partial \psi}{\partial x_l}(\underset{\sim}{q}^k) - u_h N^k \frac{\partial u_h}{\partial x_l})\, dx$$

$$T^{k,l}_{h2} = -\cos\theta_c \int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k (1 + |\nabla \underset{\sim}{\psi}|^2)^{1/2})\, dx$$

$$I_{h3}^{k,l} = \int_{\Omega_h} \frac{1}{\mathscr{E}_h^{1/2}} (\underset{\sim}{\nabla} (N^k X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k)) - \underset{\sim}{\nabla} (N^k \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k))) \cdot \underset{\sim}{\nabla} p_h \, dx$$

$$- \int_{\Omega_h} \frac{1}{\mathscr{E}_h^{3/2}} ((\underset{\sim}{\nabla} u_h \cdot \underset{\sim}{\nabla} (N^k \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k))) - (\underset{\sim}{\nabla} u_h \cdot \underset{\sim}{\nabla} (N^k X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k)))) (\underset{\sim}{\nabla} u_h \cdot \underset{\sim}{\nabla} p_h) \, dx$$

$$- \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h}{\mathscr{E}_h^{1/2}} \cdot \underset{\sim}{\nabla} (N^k \frac{\partial p_h}{\partial x_l} (\underset{\sim}{q}^k)) \, dx - \int_{\Omega_h} \frac{\partial}{\partial x_l} ((\frac{\underset{\sim}{\nabla} u_h \cdot \underset{\sim}{\nabla} p_h}{\mathscr{E}_h^{1/2}}) N^k) \, dx$$

$$- bo \int_{\Omega_h} N^k (X_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l} (\underset{\sim}{q}^k)) p_h \, dx + bo \int_{\Omega_h} u_h N^k \frac{\partial p_h}{\partial x_l} (\underset{\sim}{q}^k) dx$$

$$- bo \int_{\Omega_h} \frac{\partial}{\partial x_l} (u_h N^k p_h) \, dx - \frac{1}{\epsilon_A} \left[\int_{\Omega} N^k (x_h \frac{\partial \psi}{\partial x_l} (\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l}) f \, dx \right] \left[\int_{\Omega} f p_h dx \right]$$

$$+ \frac{1}{\epsilon_A} \left[\int_{\Omega} f (u_h - \psi) dx - vol \right] \left[\int_{\Omega_h} f \, N^k \frac{\partial p_h}{\partial x_l} (\underset{\sim}{q}^k) \, dx \right]$$

$$- \frac{1}{\epsilon_A} \left[\int_{\Omega_h} f (u_h - \psi) dx - vol \right] \left[\int_{\Omega_h} \frac{\partial}{\partial x_l} (N^k f \, p_h) \, dx \right]$$

$$- \frac{1}{\epsilon_A} \left[\int_{\Omega_h} \frac{\partial}{\partial x_l} (f \, (u_h - \psi) N^k dx \right] \left[\int_{\Omega_h} p_h \, f \, dx \right]$$

$$-\frac{1}{\epsilon_B}\int_{\Omega_h} MN^k(X_h \frac{\partial\psi}{\partial x_l}(\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l}(\underset{\sim}{q}^k))\, p_h\, dx$$

$$+\frac{1}{\epsilon_B}\int_{\Omega_h} H(u_h-\psi)\, N^k \frac{\partial p_h}{\partial x_l}(\underset{\sim}{q}^k)\, dx - \frac{1}{\epsilon_B}\int_{\Omega_h} \frac{\partial}{\partial x_l}(H(u_h-\psi)\, N^k\, p_h)\, dx$$

Proof : $J^h(\Omega_h) = \int_{\Omega_h} (1+|\underset{\sim}{\nabla} u_h|^2)^{1/2} dx + \frac{bo}{2}\int_{\Omega_h} (u_h^2 - \psi^2)\, dx$

$$-\cos\theta_c \int_{\Omega_h} (1+|\nabla\psi|^2)^{1/2} dx + A_{0h}$$

$$\therefore \delta J^h(\Omega_h) = \frac{\partial J^h}{\partial \underset{\sim}{q}^k} . \delta \underset{\sim}{q}^k = \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} \delta u_h}{g_h^{1/2}}\, dx + bo \int_{\Omega_h} u_h\, \delta u_h\, dx$$

$$+ \int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(N^k g_h^{1/2})\, dx + \frac{bo}{2}\int_{\Omega_h} \delta \underset{\sim}{q}^k . (\underset{\sim}{\nabla}(N^k u_h^2) - \underset{\sim}{\nabla}(N^k \psi^2))dx$$

$$-\cos\theta_c \int_{\Omega_h} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(N^k(1+|\underset{\sim}{\nabla}\psi|^2)^{1/2}\, dx$$

$$\Longrightarrow \frac{\partial J^h}{\partial \underset{\sim}{q}^k}\, \delta \underset{\sim}{q}^k = \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h . \underset{\sim}{\nabla} \delta \tilde{u}_h^*}{g_h^{1/2}}\, dx + \int_{\Omega} bo\, u_h\, \delta \tilde{u}_h^*\, dx + \int_{\Omega} \delta \underset{\sim}{q}^k . \underset{\sim}{\nabla}(N^k g_h^{1/2})dx$$

$$+ \frac{bo}{2}\int_{\Omega_h} \delta \underset{\sim}{q}^k . (\underset{\sim}{\nabla}(N^k u_h^2) - \underset{\sim}{\nabla}(N^k \psi^2))dx + \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h}{g_h^{1/2}} . [\underset{\sim}{\nabla}(N^k X_h \underset{\sim}{\nabla}\psi_{\Gamma_h}(\underset{\sim}{q}^k) . \delta \underset{\sim}{q}^k$$

$$-\nabla(N^k \nabla u_h(\underset{\sim}{q}^k).\delta \underset{\sim}{q}^k)]dx$$

$$+bo\int_{\Omega_h} u_h N^k (X_h \underset{\sim}{\nabla}\psi_{\Gamma_h}(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k - \underset{\sim}{\nabla} u_h(\underset{\sim}{q}^k).\delta\underset{\sim}{q}^k)\, dx$$

$$-\cos\theta_l \int_\Omega \delta\underset{\sim}{q}^k.\underset{\sim}{\nabla}(N^k(1+|\underset{\sim}{\nabla}\psi|^2)^{1/2})dx$$

$$\therefore\ \frac{\partial J^h}{\partial q_l^k} = \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(\theta_{hl}^k)}{\mathcal{E}_h^{1/2}}\, dx + \int_\Omega bo\ u_h\ \theta_{hl}^k\ dx$$

$$+\int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h}{\mathcal{E}_h^{1/2}}.(N^k(X_h \underset{\sim}{\nabla}\frac{\partial\psi}{\partial x_l}(\underset{\sim}{q}^k) - \underset{\sim}{\nabla}(N^k \frac{\partial u_h}{\partial x_l}(\underset{\sim}{q}^k))\, dx$$

$$+\int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k \mathcal{E}_h^{1/2})\, dx + \frac{bo}{2}\int_{\Omega_h} \frac{\partial}{\partial x_l}(N^k u_h^2 - N^k\psi^2)\, dx$$

$$+bo\int_{\Omega_h} u_h N^k (X_h \frac{\partial\psi}{\partial x_l}(\underset{\sim}{q}^k) - \frac{\partial u_h}{\partial x_l}(\underset{\sim}{q}^k))\, dx \quad - \cos\theta_c \int_\Omega \frac{\partial}{\partial x_l}(N^k(1+|\underset{\sim}{\nabla}\psi|^2)^{1/2})\, dx$$

$$= \int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(\theta_{hl}^k)}{\mathcal{E}_h^{1/2}} + bo\int_{\Omega_h} u_h\ \theta_{hl}^k\ dx + T_{h1}^{k,l} + T_{h2}^{k,l}$$

Substituting θ_{hl}^k in place of v_h in (3.22) we get

$$\int_{\Omega_h} \frac{\underset{\sim}{\nabla} u_h.\underset{\sim}{\nabla}(\theta_{hl}^k)}{\mathcal{E}_h^{1/2}}\, dx + bo\int_{\Omega_h} u_h\ \theta_{hl}^k\ dx$$

$$= \int_{\Omega_h} \left[\frac{\nabla \theta_{hl} \cdot \nabla \rho_h}{\sigma_h^{1/2}} - \frac{1}{\sigma_h^{3/2}} \langle \nabla u_h \cdot \nabla \theta_{hl}^k \rangle \langle \nabla u_h \cdot \nabla \rho_h \rangle \right] dx \,,$$

$$+ \int_{\Omega_h} bo \; \theta_{hl}^k \; \rho_h \; dx + \frac{1}{\epsilon_A} \left(\int_{\Omega_h} \theta_{hl}^k \; f \; dx \right) \left(\int_{\Omega_h} \rho_h \; f \; dx \right) \qquad (3.24)$$

Now we invoke Corollary 3.1 to replace the R.H.S. of (3.24) and also substituting ρ_h in place of v_h in the Corollary 3.1 we get (3.23)

3.6 DESCRIPTION OF THE NUMERICAL SCHEME : -

Let G^j (j = 1,2 ......,no. of boundary nodes) denote the function describing the position of the interior nodes of T_h w.r.t the boundary nodes.

i.e. $q^k = G^k(\{q^l\}) \;\forall\; q^k \in K^c$ and $q^l \in K$.

Now we can obtain the expression for the derivative of the energy functional as follows : -

$$\frac{\partial J_h}{\partial q_j^k} = \frac{\partial J_h}{\partial q_j^k} \Bigg|_{q^i} + \sum_{q^i \in K^c} \frac{\partial J_h}{\partial q_j^k} \frac{\partial G^i}{\partial q_j^k} \quad [j = 1,2,\ldots\ldots n) \qquad (3.25)$$

The algorithim is as follows

(1) Choose $\Omega_h^{(0)}$ [initial mesh], nmax, eptol

do(.)m = 1, nmax

(2) Compute $u_h^{(m)}$ from (P_{hl})

(3) Compute $\rho_h^{(m)}$ from (3.22)

(4) Compute the gradient $g_i = -\dfrac{\partial J_h}{\partial q_i^l}$; $\underset{\sim}{q}^l \in K$: $i = 1, 2, \ldots\ldots, n$ from (3.25).

(5) Compute the approximation of $\rho^{(m)}$ of $\underset{\forall\, \underset{\sim}{q}^l \in K}{\arg\min}\; J_h \{\underset{\sim}{q}^{(m)l} + \rho \underset{\sim}{g}^{(m)}\}$

(6) Update the mesh Compute $\Omega_h^{(m+1)}$ by $\underset{\sim}{q}^{(m+1)l} = \underset{\sim}{q}^{(m)l} + \underset{\sim}{\rho}^{(m)} \underset{\sim}{g}^{(m)}$
$\underset{\sim}{\forall}\ \underset{\sim}{q}^l \in K$; $\underset{\sim}{q}^{(m+1)} = G^k(\{q^l\})$

(7) if $\sum_{i=1}^{n} (g_i^l)^2 <$ *eptol stop*

(.) Continue

3.7 MODEL PROBLEM : -

We consider a simple problem of the determination of the equilibrium-surface of a liquid in a conical-vessel.

The following properties of the system are used.

$\sigma_l = 1.0$ $\quad\theta_c = 30^o$

$\sigma_s = 0.001$ $\quad H = 100$

$\sigma_g = 0.8661$ $\quad vol = 100$

$\alpha = 45^o$

[where α is the angle of the cone and H the height [Fig. 3.4a]]

3.7.1 Axi-symmetric Formulation :

The problem is an axi-symmetric one and hence it is reducible to a 1-dimensional problem involving the radius as the

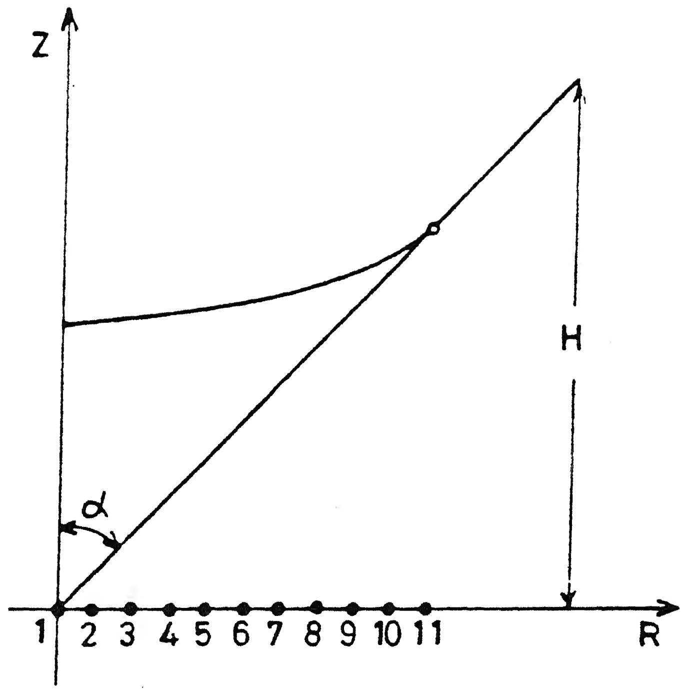


FIG 3.4a FINITE-ELEMENT DESCRIPTION OF THE MODEL PROBLEM

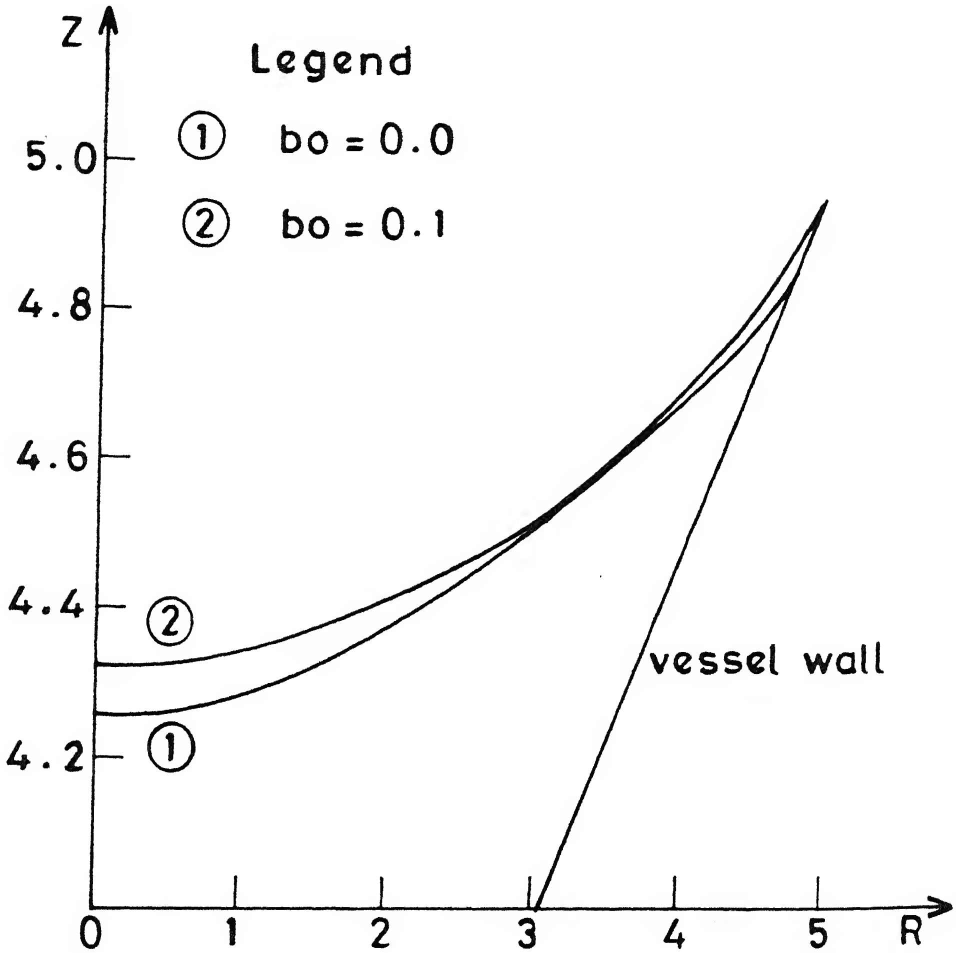


FIG.3.4b THE LIQUID SURFACE FOR DIFFERENT bos.

variable. The finite-element discretization is performed with the help of one dimensional linear elements. In this case the mesh is kept uniform in which the coordinates of an internal node is directly proportional to its node number and hence the function relating the boundary nodes to the interior nodes turns out to be simple. The figure with finite-element discretization is shown in Fig 3.4a

The following 2 cases have been studied for the axi-symmetric problem.

(i) $bO = 0$.

The abscissa (R_c) of the contact point can be determined using the analytical formula given below [Myskis [1987]].

$$R_c = \left(\frac{3\ \mathrm{vol}}{\pi \cot^2\alpha}\right)^{1/3} \Big/ \{1 - \frac{\cot\alpha}{\sin^3(\alpha-\theta_c)}(1-\cos(\alpha-\theta_c))^2(2+\cos(\alpha-\theta_c))\}^{1/3}$$

$$= 4.92$$

This agrees with the results obtained using shape-optimization technique.

The Z-coordinates of the computed surface is given in Table 3.1.

The value of the gradient of the energy functional has been determined from (3.25) at various points and it matches with the one determined by the difference scheme.

(ii) $bo = 0.1$

The Z-coordinates of the computed surface is given in Table 3.2.

In this case also the gradient of the energy functional matches with the one determined by the difference method.

The solution surfaces for the two cases is shown in Fig. 3.4b.

Table 3.1

Node No.	R-abscissa	z- Coordinates of free surface
1	0.000E00	0.426E01
2	0.492E00	0.427E01
3	0.984E00	0.429E01
4	0.147E01	0.432E01
5	0.196E01	0.376E01
6	0.246E01	0.443E01
7	0.295E01	0.450E01
8	0.344E01	0.458E01
9	0.396E01	0.468E01
10	0.442E01	0.479E01
11	0.492E01	0.492E01

Table -3.2

Node No.	R-abscissa	Z-ordinate
1	0.000E00	0.432E01
2	0.488E00	0.433E01
3	0.976E00	0.434E01
4	0.146E01	0.436E01
5	0.195E01	0.440E01
6	0.244E01	0.444E01
7	0.292E01	0.450E01
8	0.341E01	0.457E01
9	0.390E01	0.466E01
10	0.439E01	0.476E01
11	0.488E01	0.488E01

3.7.2 Two-Dimensional Formulation

We now solve the problem described in (i) in 2-Dimensions . From the consideration of symmetry the computational domain consists of a quadrant of a circle. The initial mesh is depicted in Fig. 3.5. There are 31 nodes and 45 elements. The final solution surface is depicted in Fig.3.6. In this case the function relating the boundary nodes with the internal nodes is more complicated. In the present case the axi-symmertic property o the problem is exploited and the nodes are constrained to move along the radial directions. This considerably simplifies the data structure to be developed. The limitation on the node movements is put only from the consideration that the triangles should not be flat. The conjugate-gradient method has been used for the problem.

3.8 DISCUSSIONS :-

An alternative method of the solution of the equilibrium surface problem has been obtained on the basis of `shape-optimization' theory. The major advantage of the method is that an accurate and inexpensive technique has been developed to compute the gradient of the non-linear functional. Hence it has become possible to use an efficient optimization algorithm based on the gradient for the numerical solution. In our proposed scheme the convergence is obtained in the sense of the optimization algorithms. It is to be noted that the technique for

the gradient computation is fairly general and can be applied to other areas of mechanics involving constrained variational functionals. As an example we may cite the `Constrained-obstacle' problem which is of special interest to the researchers.

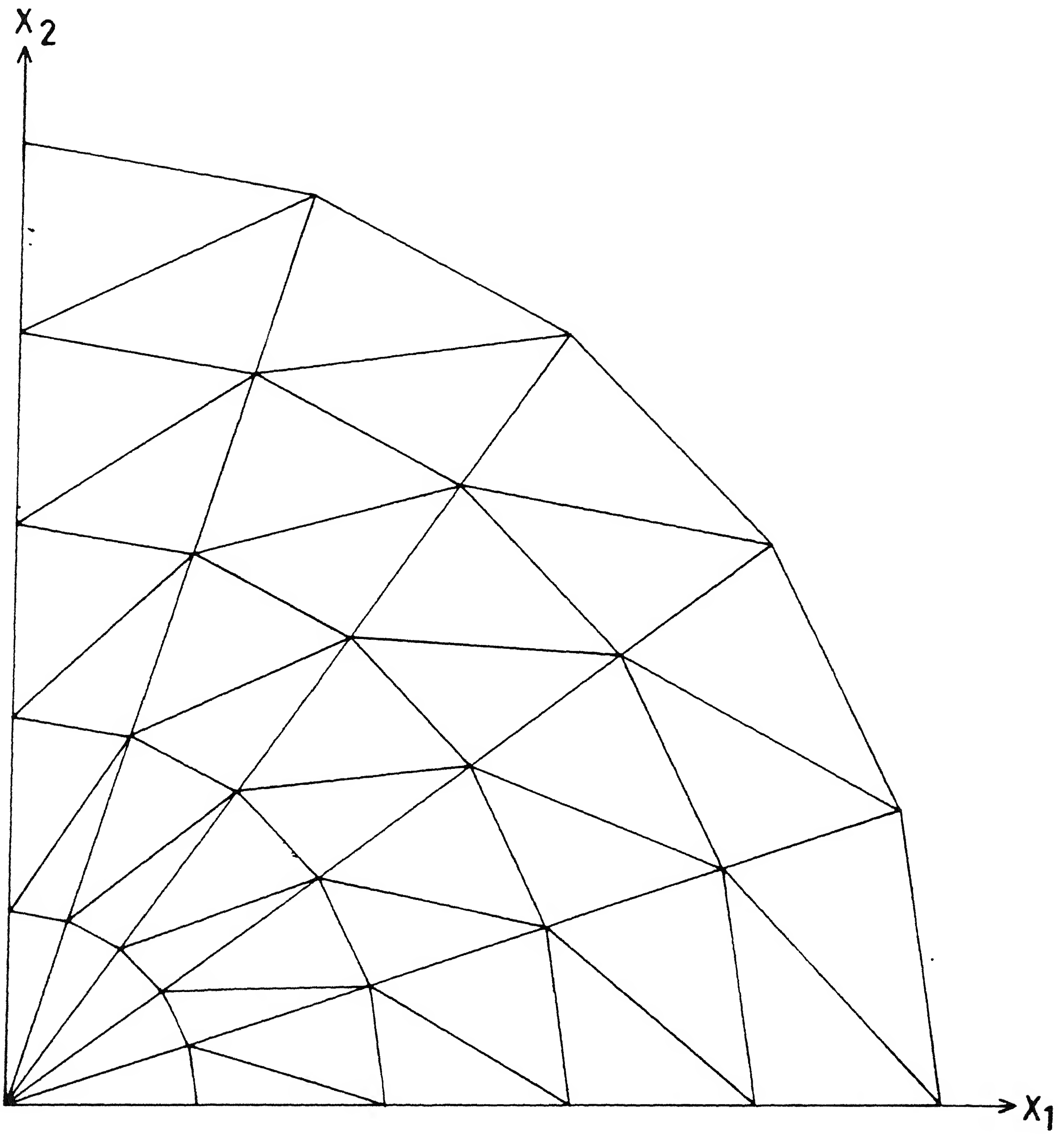


FIG. 3.5 2-D FINITE ELEMENT MESH OF THE LIQUID DOMAIN.

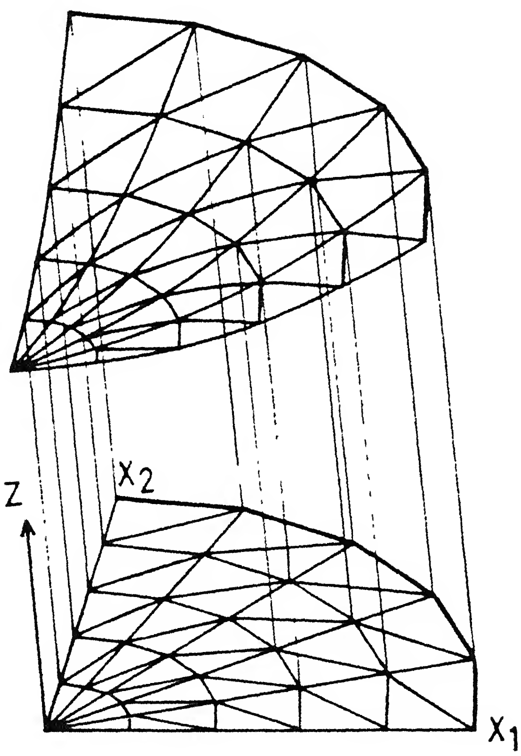


FIG. 3.6 CAPILLARY SURFACE
FOR 2-D PROBLEM.

CHAPTER IV

A FINITE ELEMENT ADAPTIVE STRATEGY FOR THE CAPILLARY SURFACE PROBLEM

4.1 INTRODUCTION

This chapter describes an h-version adaptive strategy for the finite-element scheme and the computer simulation of the capillary surface problem. An error indicator criterion based on the residual error estimate is derived and verified numerically. A suitable alternative error indicator has been proposed for adaptive purposes. A model problem is solved using the two different error indicators.

4.2 BACKGROUND AND LITERATURE SURVEY :

The shape of the capillary surface is governed by the "Laplace-Young" equation. When the domain Ω is fixed the problem is equivalent to that solution of a non-linear partial differential equation in two dimensions. Some amount of work has been done on the computational aspect of the problem. In the early stages finite-difference methods have been used by many researchers [Concus [1968],Siekmann [1981 a,b]].

As the capillary surface equation is the Euler equation corresponding to the energy functional, one can apply the variational finite element method for its solution. The finite element method has been used by Brown [1979], Orr [1975, 1977] and Mittelmann [1977]. In many cases the solution contains sharp gradients and may be inaccurate in those regions. So there is a necessity for improving the accuracy of the solutions. In the present work an adaptive scheme is proposed to solve this problem.

The general objective of the adaptive refinement scheme is to automatically change the mesh , or the approximating structure of the computational methods so as to improve the quality of the solution. Implicit in this goal is the resolution of two basic issues: the first one is to obtain a criterion for the assessment of the quality of the solution and the second one is adapting another mesh to improve quality.

The first question is resolved by obtaining an estimate of the error of the solution in some appropriate norm. Since the exact solution is in general not known a priori an attempt has been made to estimate it by deriving a-posteriori error estimates. The error estimation based on the estimates of the residuals is used widely in finite-element analysis. The residual error estimates were introduced by Babuska and Rheinboldt [1978 a,b] for linear elliptic problems and have been applied by many authors. But for the non-linear case it cannot always be guaranteed that the error is bounded by the residual. That is the reduction of the residual in some given norm does not necessarily imply the reduction in the error of the solution. However, for certain classes of non-linear operators the error is bounded by the residual. For monotone operators it is well known that the error is bound by the residual [Oden [1986a]]. In this Chapter it is shown that the differential operator of the problem is strongly monotone and hence the error is bounded.

As the estimate of the error is global in nature, the question arises as to how it can be used per se as a local refinement criteria. In general, the global estimates have been used as a basis of local enrichment of solutions. For the

refinement process to be efficient there is a need of a reliable local error indicator. It has been found that the error indicator based on the residual error estimate does not function satisfactorily due to the nature of the operator associated with the problem. But one can derive some other estimates based on the residual error which can perform as better refinement index which will depend on the nature of operators one deals with. In the present case a new error indicator has been derived which performed in a better manner as a local refinement criterion.

The second issue is related to different adaptive schemes which are a subject of much research. For finite-element approximations three broad categories of adaptive schemes are in use: In `p methods' the local degree p of the polynomial of the element shape-functions is increased to obtain a better approximation [Babuska [1981 a,b]]. The most common method is the `h-method' in which the mesh size h is reduced by successively subdiving an initial mesh thereby reducing the solution error [Babuska [1978 a,b], Oden [1986]]. The third type is the moving mesh method in which a given mesh with a fixed number of elements is fixed and the nodes are shifted so that the local errors are reduced. In the present problem the h-method is adopted.

Four-noded rectangular elements have been used for the numerical study. The data structure has been developed in which the original element is subdivided into 4 new elements [Oden [1986 b]]. As the governing equation is a non-linear one the finite-element formulation leads to a set of non-linear algebraic equations. The `full' Newton's method with block-storage scheme for the numerical solution is adopted.

4.3 PRELIMINARIES

Let $\Omega \subset\subset \mathbb{R}^2$ be an open set and Γ its boundary which is Lipschitz.

Let V be a space of functions defined as follows :-

$V := H^2(\Omega) \cap W^{1,\infty}(\Omega)$

Let us represent the elevation of the surface by the function $u(x,y)$ and let θ_c be the angle of contact [see Fig.4.1].

In this system the equations governing the capillary surface are the following

$$\underset{\sim}{\nabla}\left(\frac{\underset{\sim}{\nabla} u}{\sqrt{1+|\underset{\sim}{\nabla} u|^2}}\right) - bo\, u = q \qquad \text{in } \Omega \qquad (4.1)$$

$$\frac{\underset{\sim}{\nabla} u \,.\, \underset{\sim}{n}}{\sqrt{1+|\underset{\sim}{\nabla} u|^2}} = \cos\theta_c \qquad \text{on } \Gamma \qquad (4.2)$$

where $\underset{\sim}{n}$ denotes the outer normal to Ω and q, bo are known constants. $bo \geq 0$. For zero gravity $bo = 0$.

The variational formulation for (4.1) and (4.2) can be written as follows [Myskis [1987]] :-

Find $u \in V$ s.t.

$$\langle r, v \rangle = \int_\Omega \left(\frac{\underline{\nabla u}.\underline{\nabla v}}{\sqrt{1+|\underline{\nabla u}|^2}} + bo\, uv + qv \right) - \int_\Gamma \cos\theta_c\, v = 0 \quad (4.3)$$

$$\forall v \in V$$

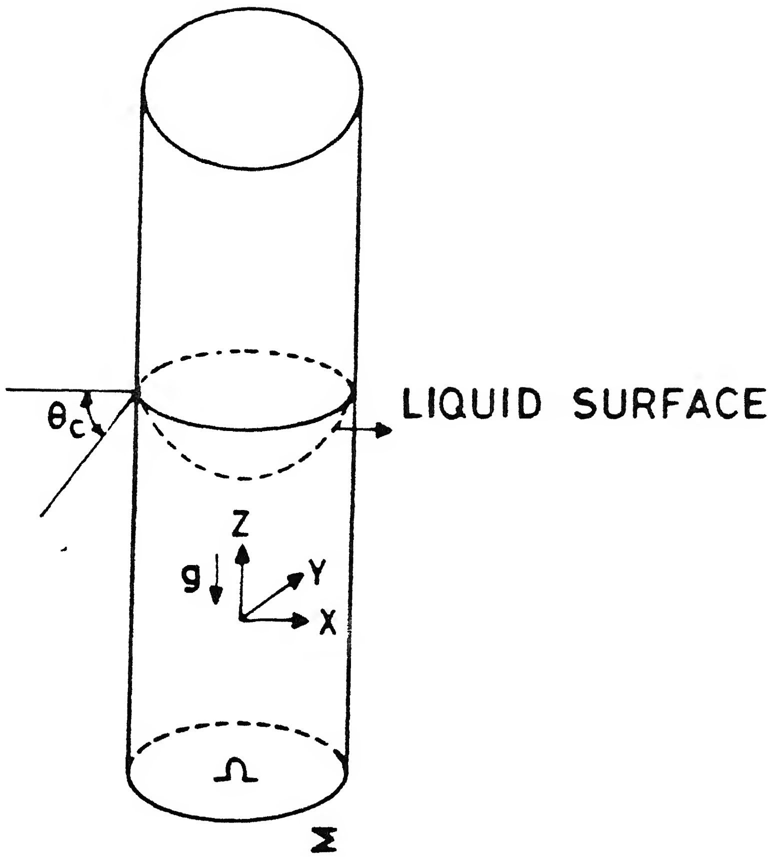


FIG.4.1 NOTATIONS

We now consider the discretization of (4.3) using finite-elements. Let $\mathcal{T}_h$ be a triangulation composed of Q' (bilinear quadrilateral) elements K, $K \in \mathcal{T}_h$ in such a way that all the vertices situated on the boundary Γ_h of the set $\overline{\Omega}_h = \bigcup_{K\in\mathcal{T}_h} K$ also belong to the boundary. With such a triangulation let us define the space $V_h' \subset V$ as follows

$$V_h' = \{ u_h \in H^2(\Omega_h) \cap W^{1,\infty}(\Omega_h) : u_h\big|_K \in Q'(K)\}$$

From the imbedding theorem we know that $H^2(\Omega) \longrightarrow C^o(\Omega)$. Hence the interpolation is well defined.

Thus for the discretized problem we have :-

Find $u_h \in V_h'$ s.t.

$$< r_h \, , \, v_h > = \int_{\Omega_h} \left[\frac{\underline{\nabla} u_h \cdot \underline{\nabla} v_h}{\sqrt{1+|\underline{\nabla} u_h|^2}} + bo\, u_h\, v_h + q\, v_h \right] - \int_{\Gamma_h} cos\theta_c\, v_h = 0 \qquad \forall\, v_h \in V_h' \tag{4.4}$$

4.4 A-POSTERIORI ERROR ESTIMATES

Let us define the following function spaces

(i) $V_h^p = \{ v_h^p \in C^o(\Omega_h) \cap V : v_h^p\big|_K \in P_p(K),\ K \in \mathcal{T}_h \}$

where $P_p(K)$ is the space of polynomials of order $p > 1$ on K.

(ii) $V_{ho}^p = \{ v_h^p \in V_h^p$ and V_h' interpolant of $v_h^p(K) = 0 \}$

We have $\| r_h \|_{V^*} = \sup\limits_{\|v\| < 1} \langle r_h , v \rangle$

An approximation of the residual is constructed according to the following :-

$$\|r_h\|_{V^*} \leq C \, \|v_o - v_h^p\| + \sup_{\|v_h^p\| \leq 1} \langle r_h , v_h^p \rangle$$

where C is a constant, $v_o \in V$ and $v_h^p \in V_h^p$.

If h stands for the mesh size we have in general $\|v_o - v_h^p\| \approx O(h^p)$ so that it makes sense asymptotically (as $h \longrightarrow 0$) to approximate

$\sup \langle r_h , v \rangle$ by $\sup \langle r_h , v_h^p \rangle$

Let us define the error $e_h = u_h - u$ [where u is the true solution] We have from (4.4)

$$\langle r_h , e_h \rangle = \int_{\Omega_h} \left\{ \left(\frac{\nabla u_h}{\sqrt{1 + |\nabla u_h|^2}} - \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \cdot \nabla e_h + bo \, e_h^2 \right\}$$

Let $\nabla u_h = \xi \; (\xi_1 , \xi_2)^T$ and $\nabla u = \eta \; (\eta_1 , \eta_2)^T$

Define $\phi(\xi , \eta) = \left(\dfrac{\xi}{\sqrt{1 + |\xi|^2}} - \dfrac{\eta}{\sqrt{1 + |\eta|^2}} \right) \cdot \dfrac{(\xi - \eta)}{|\xi - \eta|^2}$ [where $|.|$ stands for the Euclidean norm]

We find that $\phi(0 , \eta) = \dfrac{1}{\sqrt{1 + |\eta|^2}} > 0$

We now proceed to prove that $\phi(\xi , \eta) > 0$ for $\xi \neq \eta$

$$\phi(\underset{\sim}{\xi},\underset{\sim}{\eta}) = \frac{\left[\sqrt{1+|\underset{\sim}{\eta}|^2}\ \underset{\sim}{\xi} - \underset{\sim}{\eta}\sqrt{1+|\underset{\sim}{\xi}|^2}\right]}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}} \cdot \frac{(\underset{\sim}{\xi}-\underset{\sim}{\eta})}{|\underset{\sim}{\xi}-\underset{\sim}{\eta}|^2}$$

$$= \frac{\left[|\underset{\sim}{\xi}|^2\sqrt{1+|\underset{\sim}{\eta}|^2} + |\eta|^2\sqrt{1+|\underset{\sim}{\xi}|^2}\right] - \underset{\sim}{\xi}\cdot\underset{\sim}{\eta}\left[\sqrt{1+|\underset{\sim}{\xi}|^2} + \sqrt{1+|\underset{\sim}{\eta}|^2}\right]}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}\ |\underset{\sim}{\xi}-\underset{\sim}{\eta}|^2}$$

$$\geq \frac{\left[|\underset{\sim}{\xi}|^2\sqrt{1+|\underset{\sim}{\eta}|^2}+|\underset{\sim}{\eta}|^2\sqrt{1+|\underset{\sim}{\xi}|^2}\right]-|\underset{\sim}{\xi}|\,|\underset{\sim}{\eta}|\left[\sqrt{1+|\underset{\sim}{\xi}|^2}+\sqrt{1+|\underset{\sim}{\eta}|^2}\right]}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}\ |\underset{\sim}{\xi}-\underset{\sim}{\eta}|^2}$$

$$= \frac{\left[|\underset{\sim}{\xi}|-|\underset{\sim}{\eta}|\right]\left[|\underset{\sim}{\xi}|\sqrt{1+|\underset{\sim}{\eta}|^2}-|\underset{\sim}{\eta}|\sqrt{1+|\underset{\sim}{\xi}|^2}\right]\left[|\underset{\sim}{\xi}|\sqrt{1+|\underset{\sim}{\eta}|^2}+|\underset{\sim}{\eta}|\sqrt{1+|\underset{\sim}{\xi}|^2}\right]}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}\ |\underset{\sim}{\xi}-\underset{\sim}{\eta}|^2\left[|\underset{\sim}{\xi}|\sqrt{1+|\underset{\sim}{\eta}|^2} + |\underset{\sim}{\eta}|\sqrt{1+|\underset{\sim}{\xi}|^2}\right]}$$

$$= \frac{(|\underset{\sim}{\xi}| + |\underset{\sim}{\eta}|)(|\underset{\sim}{\xi}| - |\underset{\sim}{\eta}|)^2}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}\ |\underset{\sim}{\xi}-\underset{\sim}{\eta}|^2\left[|\underset{\sim}{\xi}|\sqrt{1+|\underset{\sim}{\eta}|^2} + |\underset{\sim}{\eta}|\sqrt{1+|\underset{\sim}{\xi}|^2}\right]}$$

The equality is valid only when $\underset{\sim}{\xi} = \mu\underset{\sim}{\eta}$ for some $\mu \in \mathbb{R}$.

The only case to be seen is when $\underset{\sim}{\xi} = -\underset{\sim}{\eta}$.

In that case $\phi(\underset{\sim}{\xi},\underset{\sim}{\eta}) = \dfrac{4\,|\underset{\sim}{\xi}|^2\sqrt{1+|\underset{\sim}{\xi}|^2}}{(\ .\ )} > 0$

Now we consider the limiting case when $\underset{\sim}{\xi} \to \underset{\sim}{\eta}$.

Let $\xi_1 = \eta_1 + \rho\cos\theta$ and $\xi_2 = \eta_2 + \rho\sin\theta$.

$$\phi(\rho,\theta)$$

$$= \frac{\sqrt{1+|\underset{\sim}{\eta}|^2}\left\{(\eta_1\cos\theta+\eta_2\sin\theta)-(\eta_1\cos\theta+\eta_2\sin\theta)\left[1+\frac{2\rho(\eta_1\cos\theta+\eta_2\sin\theta}{1+|\underset{\sim}{\eta}|^2}+\frac{\rho^2}{1+|\underset{\sim}{\eta}|^2}\right]^{1/2}\right\}}{\rho\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}}$$

$$\Rightarrow \phi(\rho,\theta) = \frac{1}{\sqrt{1+|\underset{\sim}{\xi}|^2}\ \sqrt{1+|\underset{\sim}{\eta}|^2}}\left\{1-\frac{(\eta_1\cos\theta+\eta_2 sin\upsilon)}{1+|\underset{\sim}{\eta}|^2}+\epsilon(\rho)\right]$$

where $\epsilon(\rho) \longrightarrow 0$ as $\rho \longrightarrow 0$.

$$\mathcal{L}im_{\rho\to 0}\ (\phi(\rho,\theta)) = \frac{1}{\sqrt{1+|\underset{\sim}{\eta}|^2}}\left[1-\frac{\eta_1\cos\theta+\eta_2\sin\theta}{1+|\underset{\sim}{\eta}|^2}\right] > 0$$

Hence we obtain for $bo > 0$

$$< r_h, e_h > \ \geq\ \alpha|e_h|^2_{1,\Omega_h} + bo\ |e_h|^2_{0,\Omega_h} \geq c_1\ \|e_h\|^2_{1,\Omega_h}$$ [for some constant $c_1 > 0$]

$$\Rightarrow \|r_h\|_{-1;\Omega_h}\ \|e_h\|\ \geq\ c_1\ \|e_h\|^2_{1,\Omega_h}$$

$$\Rightarrow \|e_h\|_{1,\Omega_h}\ \leq\ c_2\ \|r_h\|_{-1,\Omega_h}$$

When $bo = 0$ we consider the quotient space $\tilde{V} = V/\gamma$ (γ is a constant).

We have $< r_h, e_h > \ \geq\ \alpha\ |e_h|^2_{1,\Omega_h}$.

Also from the equivalence of $\|\cdot\|_{1,\Omega_h}$ and $|\cdot|_{1,\Omega_h}$ we obtain

$$|e_h|_{1,\Omega_h}\ \leq\ \|r_h\|_{-1,\Omega_h}$$

Thus the residual gives some indication of the error in the global sense.

To estimate the residual we solve the auxilliary problem :-

Find $w_h \in V_{ho}^p$ s.t.

$$\int_K \nabla w_h . \nabla v_h^p = \int_K \left[- \underset{\sim}{\nabla} . \frac{\underset{\sim}{\nabla} u_h}{\sqrt{1+|\nabla u_h|^2}} + bou_h + q \right] v_h^p + \int_{\partial K \cap \Gamma_h} \left(\frac{\underset{\sim}{\nabla} u_h . \underset{\sim}{n}}{\sqrt{1+|\nabla u_h|^2}} - \cos\theta_c \right)$$

$$+ \frac{1}{2} \int_{\partial K \setminus \Gamma_h} \left(\frac{\underset{\sim}{\nabla} u_h . \underset{\sim}{n}}{\sqrt{1+|\underline{\nabla} u_h|^2}} - \frac{\underset{\sim}{\nabla} u_h^* . \underset{\sim}{n}}{\sqrt{1+|\underline{\nabla} u_h^*|^2}} \right) v_h^p$$

$$\forall\ v_h^p \in V_{ho}^p$$

where u_h^* is the solution from the neighbouring element.

Thus $\|r_h\|_{-1,\Omega_h} \leq C \left(\sum \int |\underset{\sim}{\nabla} w_h|^2 \right)^{1/2}$

where $C = \max_K C_k$ [$C_k > 0$].

The local error indicator is $e_k = \int_K |\nabla w_h|^2$ i.e. we can use e_k as an estimate of the local error over each element. In general reducing e_k implies a reduction in $\|r_h\|_{-1,\Omega_h}$ which in turn implies a reduction in $|u - u_h|_{1,\Omega_h}$. Although the residual norm is the global indication of error it can still be considered to be a criterion for the refinement.

4.5 THE PROPOSED ERROR INDICATOR :

The simplest strategy for adaptive refinement is to compare the error indicators of each element and refine the elements which have large errors. In general the regions of sharp gradients will have large errors manifested in the form of large

error indicators. But in the type of problems considered here the effect of the gradient is not pronounced in the error indicator. This is due to the fact

that in the boundary integral we have $\dfrac{\underline{\nabla} u \cdot \underset{\sim}{n}}{\sqrt{1 + |\underline{\nabla} u|^2}}$ whereas in

ordinary linear elliptic case it is $\underline{\nabla u} \cdot \underset{\sim}{n}$. It is clearly seen that the modulus of the error in the boundary integral tends to get normalized for large values of $|\underline{\nabla u}|$ hence the error indicator of the above kind does not satisfactorily function as an index of local refinement.

In view of this we propose a new error indicator which indicates the regions of sharp gradients in a better manner.

Let $[\![\cdot]\!]$ denote the jumps across ∂K.

We define the following :-

$$\text{(i)} \quad \langle r_{oh} , v_h^p \rangle = \int_K \left[- \underline{\nabla} \cdot \left[\frac{\underline{\nabla} u_h}{\sqrt{1 + |\underline{\nabla} u_h|^2}} \right] + b_o u_h + q \right] v_h^p$$

$$\text{(ii)} \quad \langle t_{oh} , v_h^p \rangle_{\partial K \cap \Gamma_h} = \int_{\partial K \cap \Gamma_h} \left[\frac{\dfrac{\partial u_h}{\partial \underset{\sim}{n}}}{\sqrt{1 + |\underline{\nabla} u_h|^2}} - \cos \theta_c \right] v_h^p$$

(iii) $$\langle r_h' , v_h^p \rangle = \sum_K \Big\{ \langle r_{oh} , v_h^p \rangle_K + \langle t_{oh} , v_h^p \rangle_{\partial K \cap \Gamma_h} + \frac{1}{2} \Big\langle \Big[\!\Big[\frac{\partial u_h}{\partial \underset{\sim}{n}} \Big]\!\Big] , v_h^p \Big\rangle_{\partial K \setminus \Gamma_h} \Big\}$$

We have

$$\langle r_h , v_h^p \rangle = \sum_K \Big\{ \langle r_{oh} , v_h^p \rangle_K + \langle t_{oh} , v_h^p \rangle_{\partial K \cap \Gamma_h} + \frac{1}{2} \Big\langle \Bigg[\!\!\Bigg[\frac{\dfrac{\partial u_h}{\partial \underset{\sim}{n}}}{\sqrt{1 + |\underline{\nabla}\, u_h|^2}} \Bigg]\!\!\Bigg] , v_h^p \Big\rangle_{\partial K \setminus \Gamma_h} \Big\}$$

Since $$\left| \Bigg[\!\!\Bigg[\frac{\dfrac{\partial u_h}{\partial \underset{\sim}{n}}}{\sqrt{1 + |\underline{\nabla}\, u_h|^2}} \Bigg]\!\!\Bigg] \right| \le \left| \Big[\!\Big[\frac{\partial u_h}{\partial \underset{\sim}{n}} \Big]\!\Big] \right| .$$

We have

$$\langle r_h , v_h^p \rangle \le \Big(\sum_K \| s_K \|_{-1,K} \Big) \, \| v_h^p \|_{V_h^p}$$

where $$\| s_K \|_{-1,K} = \| r_{oh} \|_{-1,K} + C_{1K} \| t_{oh} \|_{0,\partial K \cap \Gamma_h} + C_{2K} \left\| \Big[\!\Big[\frac{\partial u_h}{\partial \underset{\sim}{n}} \Big]\!\Big] \right\|_{0,\partial K \setminus \Gamma_h}$$

It is possible to compute the constants C_{1K} and C_{2K} and hence

$\| e_K \|_{-1,K}$ can be determined. However, due to the computational difficulties we use the following simpler error estimate suitable for practical purposes.

We again define an auxilliary problem :-

Find $\phi_K \in V^p_{ho}(K)$ s.t.

$$\int_K \underline{\nabla}\,\phi_K \cdot \underline{\nabla}\, v^p_h = \langle r_{oh}, v^p_h \rangle_K + \langle t_{oh}, v^p_h \rangle_{\partial K \cap \Gamma_h} + \frac{1}{2} \left\langle \underline{\left[\!\left[\frac{\partial u_h}{\partial \tilde{n}} \right]\!\right]}, v^p_h \right\rangle_{\partial K \setminus \Gamma_h}$$

$$\forall\, v^p_h \in V^p_h$$

Therefore we have

$$\langle r'_h , v^p_h \rangle = \sum_K \int_K \underline{\nabla}\,\phi_K \cdot \underline{\nabla}\, v^p_h$$

$$\Longrightarrow \| r'_h \|_{-1} \le c \left(\sum \int_K | \underline{\nabla}\,\phi_K |^2 \right)^{1/2} .$$

We notice that $\underline{\left[\!\left[\frac{\partial u_h}{\partial \tilde{n}} \right]\!\right]} \to 0$ as $h \to 0$.

$$\Longrightarrow \underline{\left[\!\left[\frac{\frac{\partial u_h}{\partial \tilde{n}}}{\sqrt{1 + |\underline{\nabla}\, u_h|^2}} \right]\!\right]} \to 0 \quad \text{as} \quad h \to 0.$$

Therefore $\| r'_h \|_{-1} \to \| r_h \|_{-1}$ as $h \to 0$.

Thus $\| r_h \|_{-1} \le c \left(\sum \int_K | \underline{\nabla}\,\phi_K |^2 \right)^{1/2}$ as $h \to 0$.

We now define our error indicator as $\tilde{e}_K = \int_K | \underline{\nabla}\,\phi_K |^2$.

The practical advantage of using $\tilde{e}_K$ is clearly demonstrated in the model problem. The corner regions near the vessel wall have sharp gradients for the solution and hence maximum refinement is done in those regions [Fig. 4.5b].

4.6 ADAPTIVE MESH STRATEGY :

The important steps in h-refinement strategy are as follows :

Step 1) Form the triangulation $\mathcal{T}_h$ out of Ω.

Step 2) Solve equation (4.4) using non-linear iteration scheme e.g. Newton's method.

Step 3) Compute the error indicator $e_k/\tilde{e}_k$ for each element K and form a refinement index $e_I = \mu + \beta\sigma$ [where μ is the mean $e_k/\tilde{e}_k$ and σ the standard deviation and $1 > \beta > 0$.]

Step 4) Obtain the solution at the new nodes using interpolation and the Lagrangian multiplier approach to take into account the constraint nodes.

Step 5) Repeat from step 2 until $e_k/\tilde{e}_k$ is small or the maximum number of elements permissible has been reached.

4.6.1 Details of Data Structure in the Computer Code :

An important consideration in all adaptive schemes is the data structure and the associated algorithm needed to handle the changing total number of elements, nodal locations, numbers

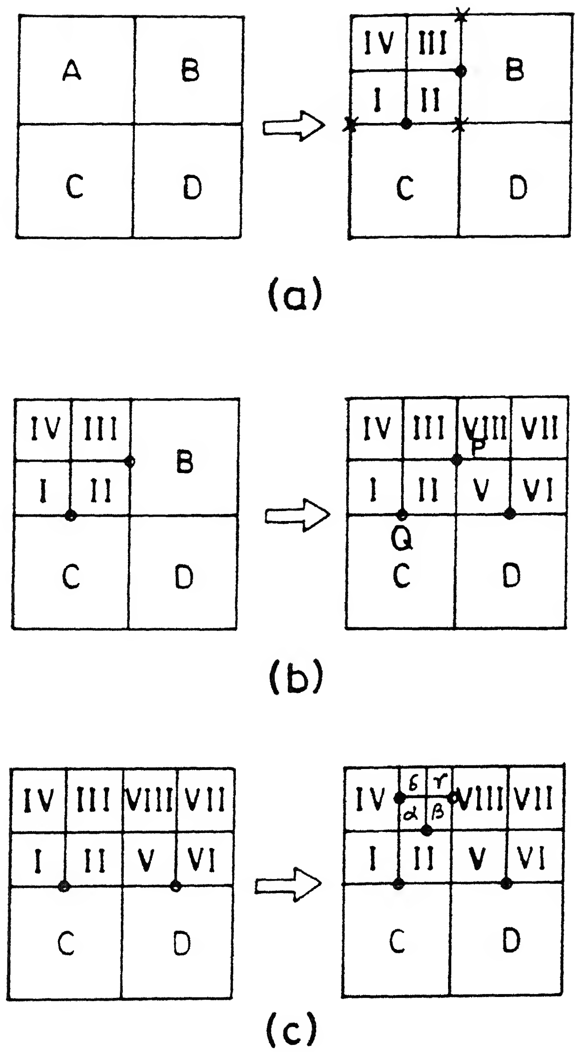


FIG.4.2 SEQUENCE OF REFINEMENTS OF A UNIFORM MESH.

and element labels. The data structure developed in the present problem is similar to the one developed by Oden et al. [1986 b]. We shall now briefly describe the data structure which has been developed:

In the present refinement process an element is refined into 4 sub-elements. The book keeping of the elements and the refinement process is monitored by the arrays NELCON (8, NEL), NELSID (8,NEL), XCOOR(NUMNP) YCOOR(NUMNP), and LEVEL (NEL) where

NUMNP is the number of nodal points in the system.

NEL is the number elements.

XCOOR is the array containing the X coordinates and

YCOOR is the array containing the Y coordinates of the nodes

NELCON is the element connectivity. It is to be noted that although we 4 noded-quadrilaterals are usedthere can be 4 additional nodes on the sides due to refinement. Thus we have to keep the information for 8 nodes in an elements.

NELSID is the array containing the information of the adjacent element.

LEVEL is the array which assigns a level number to an element. Initially, the same level is assigned to all the elements. For example the initial coarse mesh could be assigned level 1. When an element is refined the sub-elements belong to a higher level (2) result. This process can go on indefinitely in general.

The algorithm for refinement is described in the following steps.

Step 1 Loop over the neighbours of the element (which is made possible with the NELSID array) and check the level of refinement of the neighbouring elements ralative to the level of the element to be refined.

Step 2 If any neighbouring element has a level lower than the element to be refined then refinement is not posible at this state.

Step 3 If the element can be refined, we generate new element numbers and modify the necessary parameters.

Step 4 Compute the modified NELSID and NELCON arrays.

Step 5 Adapt the connectivity of the negihbouring elements

Step 6 Interpolate the solution at the new nodes.

It is clear that some strategy is needed to test if the elements are appropriately connected for the refinement to take place. Let us consider, for example, the uniform grid of four elements shown in Fig.4.2(a) and suppose that the error estimater dictates the element A is to be refined. Thus, A is divided into four elements, I.II,III,IV, as shown, and the solution values at the junction nodes, (shown circled in the figure), are constrained to coincide with the averaged values between those marked X. It may also be noted that the connectivities change in this process, e.g., the connectivities 4 and 8 of element B are different.

Next let us assume that an additional refinement is required, and that we must next refine element III. We impose the restriction that each element side cannot have more than two elements connected to it. Thus, before III can be refined,

element B must first be refined, as indicated in Fig. 4.2(b). The constrained node P in Fig. 4.2(a) now becomes active, while node Q remains a constrained node. With B bisected, we proceed to refine III into subelements $\alpha,\beta,\gamma,\delta$, and new constrained nodes, again circled in Fig. 4.2(c), are produced. In this case, only element B had to be refined first in order to refine III, but, in general, the number of elements that must be refined in order to refine a particular element cannot be specified.

4.7 NUMERICAL RESULTS

4.7.1 Numerical Experiment

We perform a numerical experiment to verify the assertion that the norms of the residual and that of the error decreases with refinement. For this purpose we take up a simple 1-D problem whose solution is known.The problem is as follows :-

$$\frac{d}{dx}\left[\frac{u_x}{\sqrt{1+u_x^2}}\right] = 1.0 \quad \text{in } \Omega_h \qquad [\ \Omega_h = (0,0.5)\ ]$$

$$\frac{d}{dx}\left[\frac{u_x}{\sqrt{1+u_x^2}}\right]\Bigg|_0 = 0$$

$$\frac{d}{dx}\left[\frac{u_x}{\sqrt{1+u_x^2}}\right]\Bigg|_{0.5} = 0.5 \qquad [\ u_x \text{ stands for } \frac{du}{dx}\ ]$$

As usual the $V_h^1(\Omega_h)$ space is composed of linear elements.

The $V_h^p(\Omega_h)$ space is composed of quadratic polynomials (i.e. p = 2). The norm of the residual is computed according to the definition

$$\|r_h\| = \sup_{|v_h|\le 1} \langle r_h, v_h \rangle$$

A constrained optimization algorithm is used for this purpose. The variation of the norms with refinement is given in Table 4.1. It can be seen that the residual norm gradual decreases with an increase in the number of elements which vindicates our assertion.

4.7.2 2-Dimensional Example

We now apply our adaptive scheme using the two error indicators to solve a 2-Dimensional problem as shown in Fig. 4.3.

The problem is to determine the capillary surface over a rectangular domain in zero gravity. $\theta_c = 45^0$ makes the gradient infinite which makes the solution difficult. The solution is arrived by two methods :

I) Refinement with error Indicator e_K :

The initial mesh (10X10)is shown in Fig. 4.4a and the final mesh after 5 refinements are depicted in Fig. 4.4b.

The variation of the error indicator norms with the number of elements is given in Table 4.2. It may be observed that gradient is high in the top right corner and indicator norm the error decreased by a factor of 10 in 5 iterations.

II) Refinement with Error Indicator $\tilde{e}_K$.

Table 4.1

No. of Elements	$\Vert r_h \Vert_{-1}$	Error Indicator Norm	$\vert u-u_h \vert_1$
4	.300 E-03	.255 E-01	.237 E-01
9	.755 E-04	.133 E-01	.118 E-01
14	.336 E-04	.729 E-02	.792 E-02
19	.189 E-04	.537 E-02	.594 E-02
24	.121 E-04	.425 E-02	.475 E-02
29	.840 E-05	.315 E-02	.396 E-02
49	.302 E-05	.208 E-02	.237 E-02

Table 4.2

Refinement	No. of Elements	Error Indicator Norm
1	100	4.230 E-02
2	130	3.944 E-02
3	196	3.539 E-02
4	400	2.154 E-02
5	487	2.012 E-02

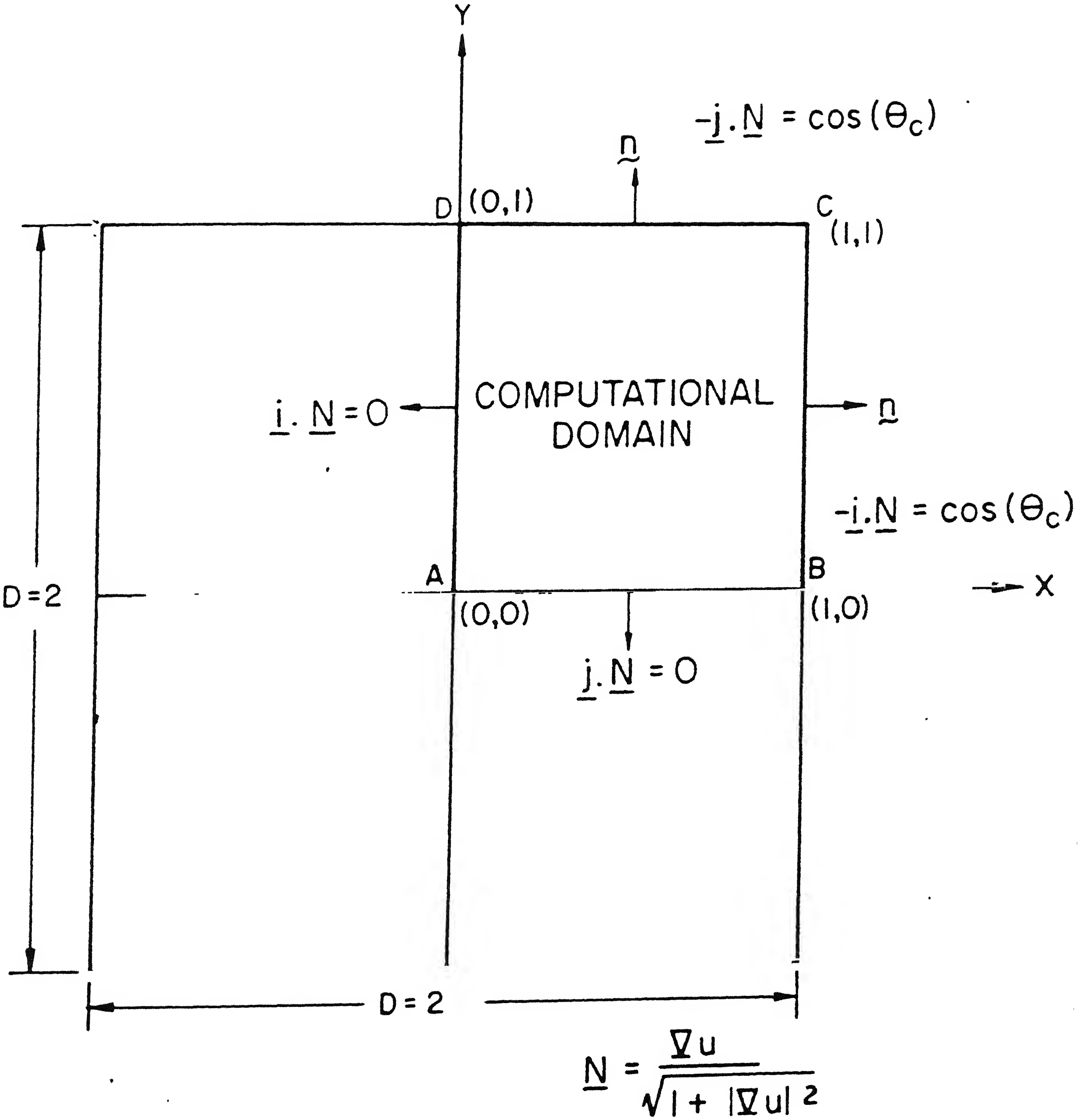


FIG. 4.3 COMPUTATIONAL DOMAIN.

The initial mesh (2x2) is shown in Fig. 4.5a and the final mesh after 5 refinements is depicted in Fig. 4.5b. The variation of the error indicator norms with the number of elements is given in Table 4.3. It may be observed that the elements are densely clustered near the top right corner which shows the effectivity of the error indicator.

4.8 DISCUSSIONS :

An a-posteriori error estimate has been developed for the non-linear problem which is also verified numerically. An alternative error indicator has been proposed whose practical advantage is also shown. The h-refinement procedure seems to be effective as the time spent in the logic of refinement is less and the solution is interpolated at the new nodes which ensures fast convergence at higher level.

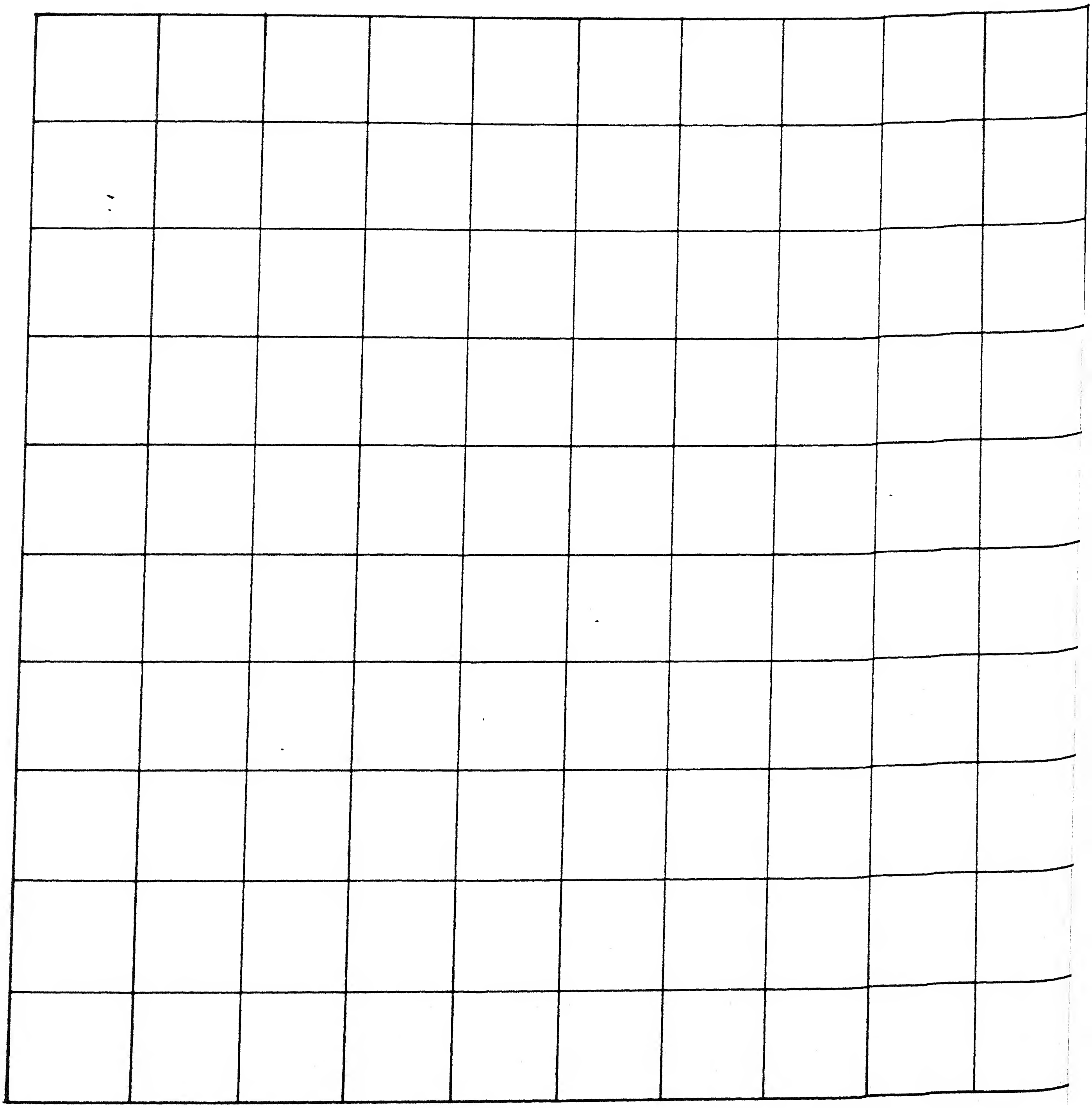


FIG. 4.4a INITIAL MESH

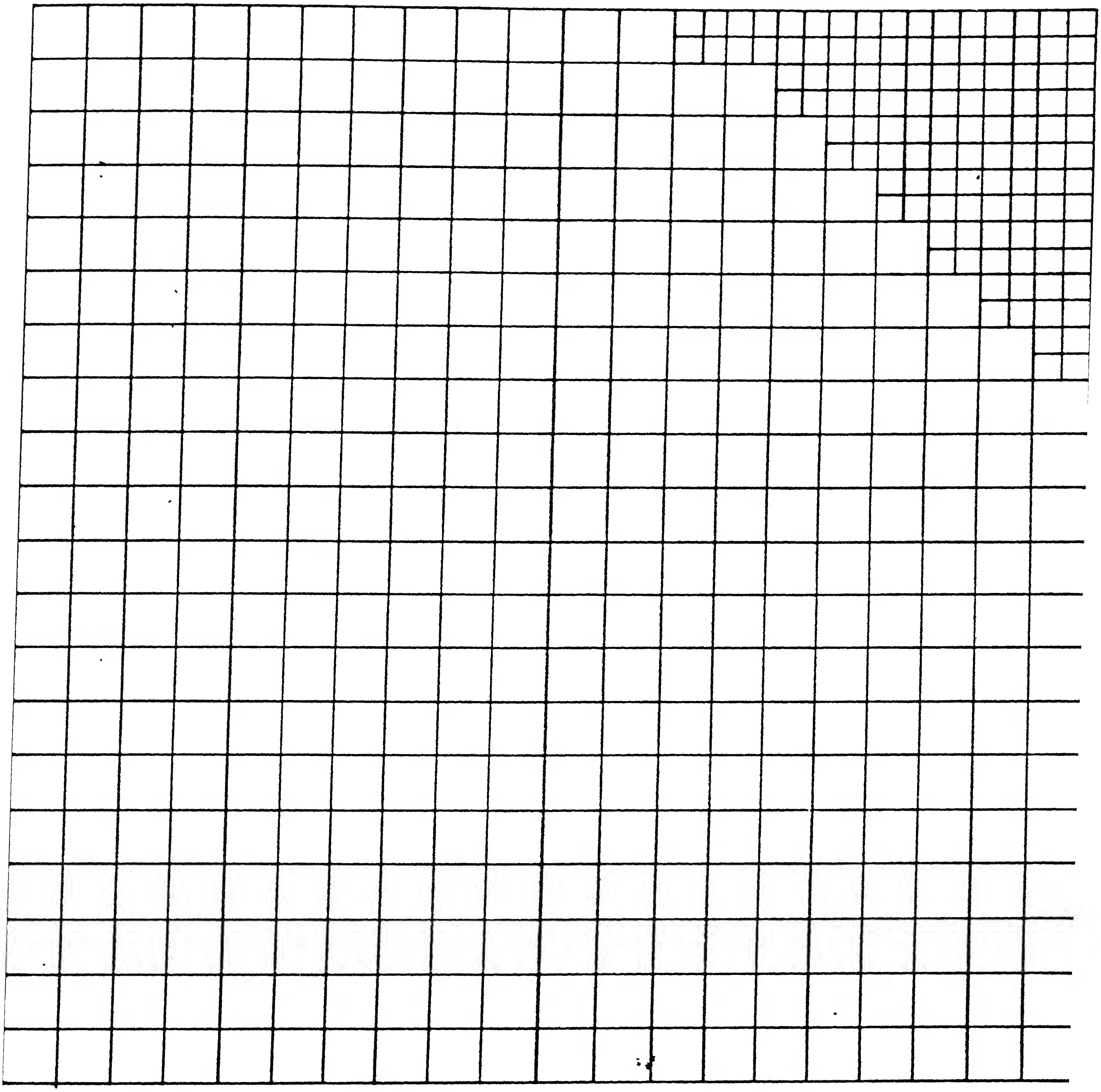


FIG. 4.4b MESH AFTER REFINEMENT WITH e_K.

Table 4.3

Refinement	No. of Elements	Error Indicator Norm
1	4	0.228 E-00
2	13	0.144 E-00
3	25	0.981 E-01
4	67	0.551 E-01
5	82	0.514 E-01
6	103	0.483 E-01
7	133	0.448 E-01
8	265	0.277 E-01
9	316	0.258 E-01
10	388	0.245 E-01

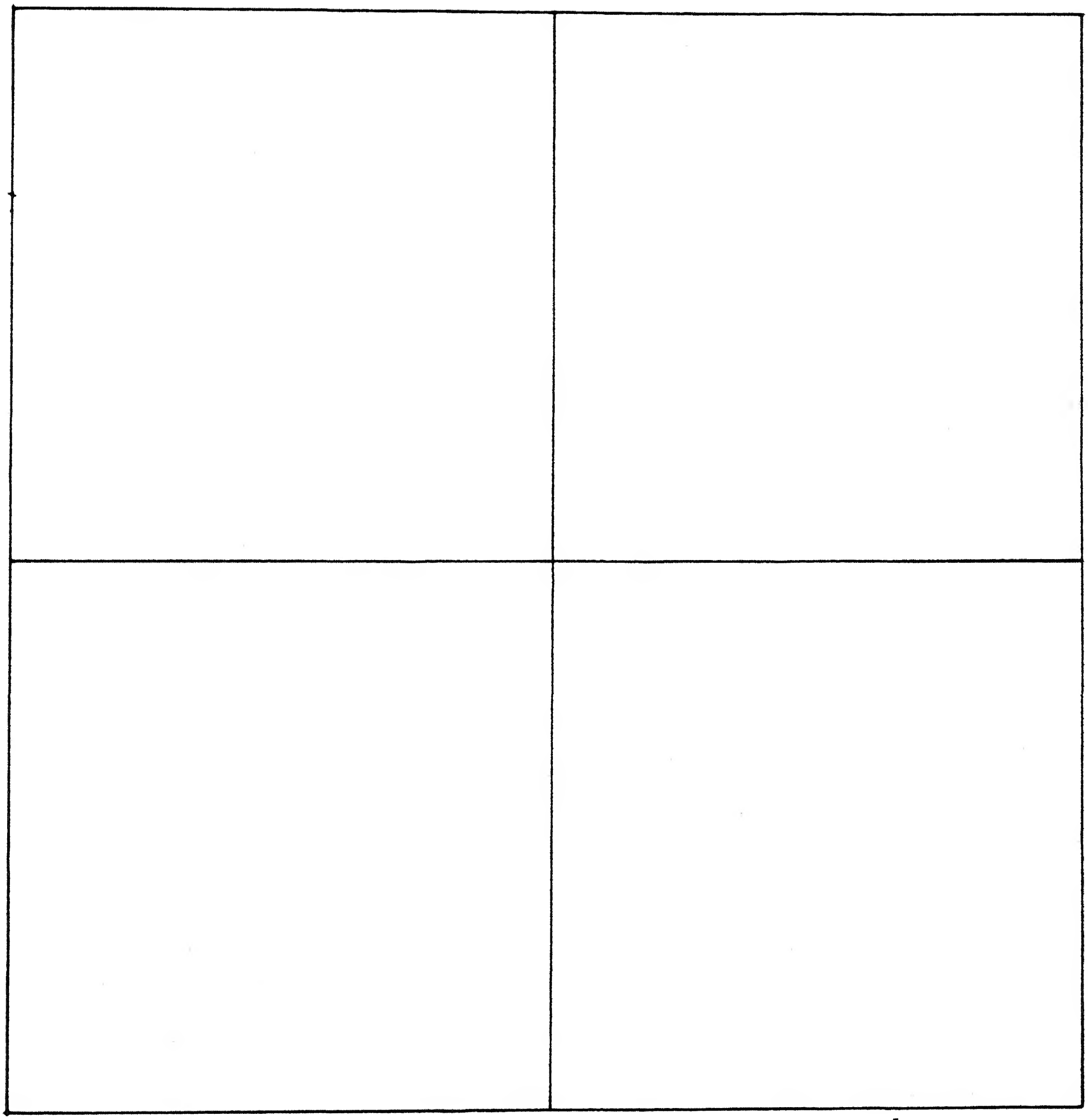


FIG. 4.5a INITIAL MESH.

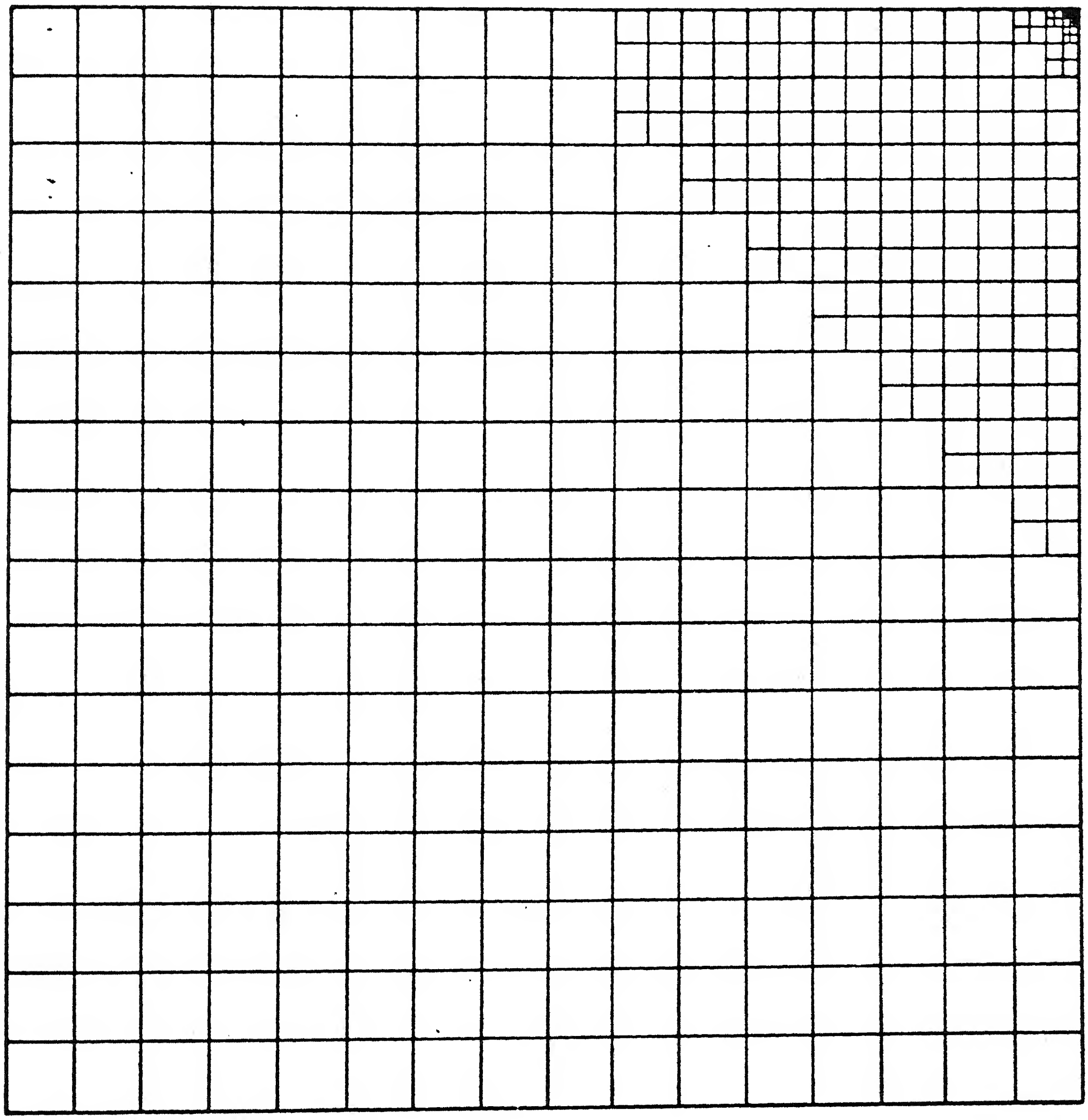


FIG. 4.5b MESH AFTER REFINEMENT WITH $\tilde{e}_K$.

CHAPTER V

OSCILLATIONS OF AN IDEAL LIQUID IN LOW GRAVITY

5.1 INTRODUCTION :

In this Chapter a description of a numerical scheme using finite elements is presented for the computation of the eigen-values of oscillations of an ideal liquid under Low gravity conditions. The major difficulty in this study is the fact that the differential operator defined on the boundary is of higher order than that of the spatial operator in the domain. Nevertheless the problem is cast into a standard eigen-value problem with a suitable choice of operators. A Finite Element procedure has been described in detail and numerical results are presented.

5.2 LITERATURE SURVEY AND GENERAL BACK-GROUND :

Linear oscillations of an ideal capillary liquid represents a classical problem in fluid mechanics. Lamb [1945] has obtained some results in this study. The study of the liquid oscillations is of special interest in space-technology. The satellite contains liquid propellants and the magnitude of the natural frequencies of the propellant and its behaviour is of prime importance for the stability investigations.

The dynamic movement of the liquid surface is termed in the scientific literature as "sloshing". A substantial amount of study has been carried out on the sloshing behaviour of the liquids under normal conditions Budiansky [1960], Bauer [1963a,b],

Levin [1963], Tong [1966], Zienkiewicz et al. [1978], Haroun [1981]. But not much work was done about sloshing in micro-gravity or zero-gravity conditions. Bauer [1981, 1984, 1989 a,b,c] adopted a semi-analytical method for the computation of the eigen-frequencies. In the work vessels with simple geometries were considered and this method becomes very difficult for the complex geometries or even in simple 3-D problems.

A comprehensive treatment of the numerical methods for the computations of the eigen-fequencies is given by Myskis [1987]. But the numerical methods covered do not include the Finite-Element Method. An outline for the construction of operators so as to reduce the problem to a standard eigen-value problem was discussed. In the present work this concept has been extended to the construction of the operators in the Finite Element spaces with a mathematical justification. Also a proof is furnished on the existence of the eigen-values with appropriately chosen function spaces.

5.3 NOTATIONS AND PRELIMINARIES :

A system comprising of a liquid, a solid and a gas is depicted in Fig. 5.1. The liquid is assumed to be the inviscid and incompressible.The region occupied by the liquid is denoted by Ω. The surface of contact between the liquid and the walls of the vessel (assumed to be rigid and smooth) is denoted by Σ and the equilibrium surface is denoted by Γ. $\partial\Gamma$ denotes the line of contact of the liquid surface with the vessel and θ_c the angle of

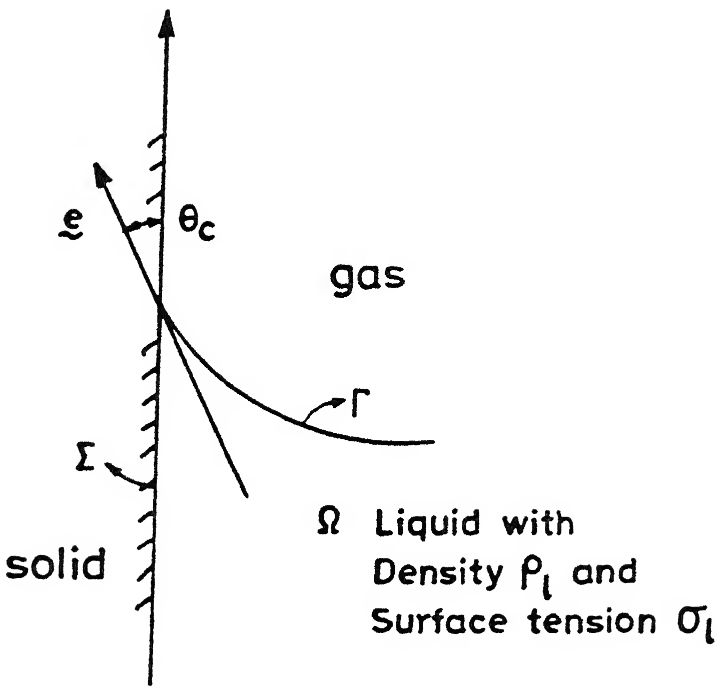


FIG. 5.1 Notations.

contact of the liquid surface with the vessel.

We define the following function spaces.

$$H^m(\Omega) = \left\{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega)\ \forall |\alpha| \le m\right\}$$

where α is a multi-index, and $m \ge 0$ is any positive integer.

$$V(\Omega) = \left\{v \in H^1(\Omega); \int_\Omega v = 0\right\}$$

$$V(\Gamma) = \left\{v \in H^1(\Gamma); \int_\Gamma v = 0\right\}$$

$\gamma_o u|_\Gamma$ = trace of u on Γ

For convenience, in further discussion we shall use u_Γ in place of $\gamma_o u|_\Gamma$.

For the details regarding the space $H^1(\Omega)$ and the trace operator γ_o one may refer to Oden [1976] or Adams [1976].

Let ϕ denote the velocity potential of the fluid and λ the eigen-value of the system and $\tilde{\lambda} = \lambda\ (\rho_l/\sigma_l)$

Governing equations in the fluid are given by : -

$$\left.\begin{aligned} &\Delta\phi = 0 \quad \text{in } \Omega \\ &\frac{\partial \phi}{\partial \underset{\sim}{n}} = 0 \quad \text{on } \Sigma \text{ (no- flow condition)} \end{aligned}\right\} \qquad (5.1)$$

where $\underset{\sim}{n}$ denotes the outward normal to Ω on Σ

The following Euler equations govern the free-surface.

$$\left.\begin{aligned} &-\Delta_\Gamma \frac{\partial\phi}{\partial \underline{n}} + bo\,\frac{\partial\phi}{\partial \underline{n}} - C = \tilde{\lambda}\,\phi_\Gamma \qquad \text{in } \Gamma \\ &\text{with the b.c.} \\ &\qquad \frac{\partial}{\partial \underline{e}}\left(\frac{\partial\phi}{\partial \underline{n}}\right) + \chi\,\frac{\partial\phi}{\partial \underline{n}} = 0 \qquad \text{on } \partial\Gamma \\ &\text{and the volume condition } \int_\Gamma \frac{\partial\phi}{\partial \underline{n}} = 0 \end{aligned}\right\} \qquad (5.2)$$

Where $bo(\xi_1,\xi_2)$ for $(\xi_1,\xi_2) \in \Gamma$ and $\chi(s)$ for $s \in \partial\Gamma$ are given functions depending on the system properties. c is a constant corresponding to the Lagrangian multiplier for the volume constraint and $\underline{e}$ is the tangent vector [see Fig 5.1]

We assume $bo \in L^\infty(\Gamma)$ and $\chi \in L^\infty(\partial\Gamma)$ with $\chi > 0$.

Δ_Γ occuring in (5.2) is the Laplace Beltrami Operator on the 2-D manifold Γ. (the details are given in [Appendix I]).

The distinguishing feature of the boundary value problem (5.1) and (5.2) is that the equations featuring in (5.2) contain derivatives of higher order than the spatial equation in (5.1) :

The present structure is not readily amenable to the variational method of solution, but can be reduced to a standard eigen-value problem with a suitable choice of operators. We shall proceed now to define the operators.

Let u denote the normal velocity of a point on Γ.

Thus $u = \dfrac{\partial\phi}{\partial \underline{n}}$.

Let us define an operator $B : L^2(\Gamma) \rightarrow L^2(\Gamma)$ as follows :-

For a given $u \in L^2(\Gamma)$ *let* ϕ denote the solution of the following problem :-

Find $\phi \in V(\Omega)$ *s.t.*

$$\int_\Omega \underline{\nabla}\phi . \underline{\nabla}v = \int_\Gamma uv \qquad (5.3)$$

$$\forall\ v \in V(\Omega)$$

Then $Bu \stackrel{def}{=} \gamma_o\ \phi|_\Gamma$

The properties of B are given in the *Lemma 5.1*

Lemma 5.1 : - B is a compact, self-adjoint and positive definite operator.

Proof :- We note that the norms $|\phi|_{1,\Omega}$ and $\|\phi\|_{1,\Omega}$ are equivalent for $u \in V(\Omega)$ [Deny [1953]]. Also the trace operator $\gamma_o : H^1(\Omega) \rightarrow L^2(\Gamma)$ is compact for $\Omega \subset \mathbb{R}^3$ [Ciarlet [1988]]. Thus B is compact. Let ϕ be a solution of (5.3) for a given $u \in L^2(\Gamma)$ and ψ a solution of (5.3) for a given $v \in L^2(\Gamma)$.

Then we have

$$\int_\Gamma (Bu)v = \int_\Omega \underline{\nabla}\ \phi . \underline{\nabla}\psi = \int_\Gamma (Bv)u.$$ Hence B is self-adjoint.

Also $\int_\Gamma (Bu)u = \int_\Omega |\underline{\nabla}\phi|^2 > 0$. Hence B is positive definite.

We define an operator $M : H^1(\Gamma) \rightarrow L^2(\Gamma)$ *as follows : -*

$$(Mu, v) = (Bu, v) = \frac{1}{2}\ [(Bu, v) + (Bv, u)]$$

We define an operator $A : H^1(\Gamma) \longrightarrow H^1(\Gamma)$ as follows :

$$(Au,v) = \int_\Gamma \nabla_\Gamma(u,v) + \int_\Gamma bou.v + \int_{\partial\Gamma} \chi uv \qquad \forall v \in H^1(\Gamma)$$

Here $\nabla_\Gamma(.,.)$ is the Beltrami first differential operator on Γ [Myskis [1987], Nitsche [1975]].

When $bo \geq 0$ clearly for $u \in V(\Gamma)$ $(Au,u) \geq \alpha \|u\|^2_{1,\Gamma}$ for some $\alpha > 0$

(5.4)

When $bo<0$ inequality (4) may not hold in general . We restrict ourself to the case in which (4) holds [11].

We define an operator $G: L^2(\Gamma) \longrightarrow V(\Gamma)$ as follows :

For any $g \in L^2(\Gamma)$, $Gg \in V(\Gamma)$ denotes the solution of the following problem :

Find $Gg \in V(\Gamma)$ s. t

$$(A(Gg),v) = (g,v) \qquad (5.5)$$

$$\forall\ v \in V(\Gamma)$$

Lemma 5.2 : G is compact as an operator from $L^2(\Gamma) \longrightarrow L^2(\Gamma)$

Proof From Rellich-Kondrasov Compactness theorem $H^1(\Gamma)$ is compactly embedded in $L^2(\Gamma)$. Hence G is compact.

We next define an operator $T = GoB : L^2(\Gamma) \longrightarrow V(\Gamma)$ as follows:

For a given $u \in L^2(\Gamma)$ let Tu be the solution of the following problem :

Find $Tu \in V(\Gamma)$ s.t.

$$(A(Tu),v) = (Bu,v) \qquad (5.6)$$

$$\forall v \in V(\Gamma)$$

Lemma 5.3 : T is compact as an operator from $L^2(\Gamma)$ to $L^2(\Gamma)$. T is also self-adjoint and positive definite.

Proof B is compact from $L^2(\Gamma)$ to $L^2(\Gamma)$, G is compact from $L^2(\Gamma)$ to $L^2(\Gamma)$.

Hence T is compact

Defining $\langle u, Tv\rangle_B = (Bu, Tv)$ we have

$\langle u,Tv\rangle_B = \int_\Gamma \underline{\nabla}_\Gamma(Tu,Tv) + \int_\Gamma b\sigma(Tu)(Tv) + \int_{\partial\Gamma}\chi(Tu)(Tv) = \langle v,Tu\rangle_B$. Hence T is self-adjoint. It is clear that $(Bu,Tu) > 0$. Hence T is positive definite.

From the theory of compact, self-adjoint operators [Taylor [1958], Yosida [1974]] we find that T has a countable number of positive eigen-values with 0 as its cluster point.

Let μ_m be the m^{th} eigen-value of T and W_m the eigen-vector.

$$TW_m = \mu_m W_m$$

$$\Rightarrow W_m = T(\mu_m^{-1} W_m)$$

$$\Rightarrow W_m = T(\lambda_m W_m) \text{ [where } \lambda_m = \mu_m^{-1}\text{]}$$

Thus λ_m is an eigen-value of the following problem :

Find $(u,\lambda) \in V(\Gamma) \times \mathbb{R}^+$ *s.t.*

$$\begin{aligned}(Au,v) &= \lambda\ (Bu,v) \\ &= \lambda\ (Mu,v) \\ &\qquad \forall v \in V(\Gamma)\end{aligned} \tag{5.7}$$

We can summarize the above results in the following theorem.

Theorem 5.1 : -

The system given by (1) and (2) has a countable number of positive eigen-values which can be arranged in a non-decreasing sequence i.e. $\{ 0 < \lambda_1 \le \lambda_2 \ldots \}$. and an orthonormal basis of eigen-functions w.r.t the inner product $(B(.),.)$.

5.4 FINITE ELEMENT FORMULATION :

For $\Omega \subset \mathbb{R}^3$ let T_Ω^h denote the triangulation of Ω into Ω_h using *3-D* finite elements.

Similarly for $\Gamma \subset \mathbb{R}^2$ let T_Γ^h denote the triangulation of Γ into Γ_h using *2-D* finite elements.

We define the following function spaces.

$$H_h^1(\Omega_h) = \{ v_h \in C^0(\Omega_h) : v_h|_K \in P_m(K) \ \forall \ K \in T_\Omega^h \}$$

$$V_h(\Omega_h) = \{ v_h \in H_h^1(\Omega_h) : \int_{\Omega_h} v_h = 0 \}$$

$$H_h^1(\Gamma_h) = \{ v_h \in C^0(\Gamma_h) : v_h|_K \in P_m(K) \ \forall \ K \in T_\Gamma^h \}$$

$$V_h(\Gamma_h) = \{ v_h \in H_h^1(\Gamma_h) : \int_{\Gamma_h} v_h = 0 \}$$

where $P_m(K)$ denotes the polynomial of order m restricted to K.

We wish to make some remarks on the modelling of the surface. Since Γ is a *2-D* manifold $\exists$ a diffeomorphism G which maps and element K on Γ to a standard finite element $\hat{K}$ in the (ξ,η) plane [Fig.5.2]. Let P denote a generic point on Γ. Then P has the following representation.

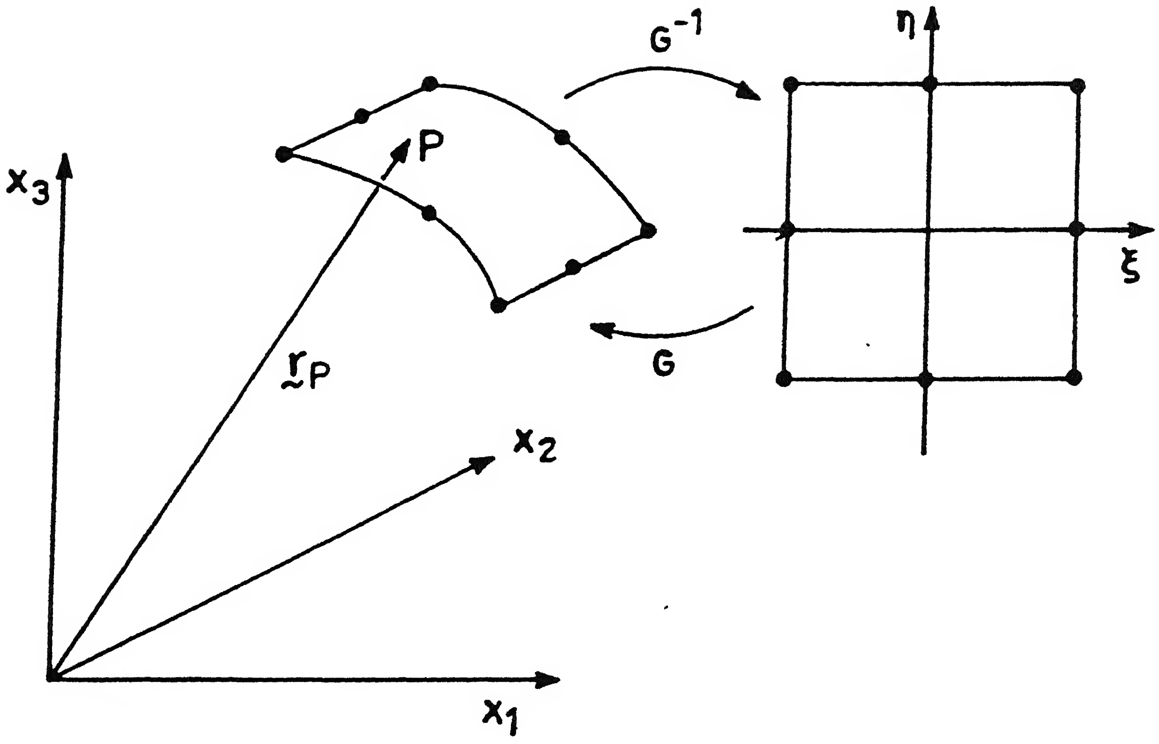


FIG. 5.2 Surface representation.

$$r_p = (x_1(\xi,\eta),\ x_2(\xi,\eta), x_3(\xi,\eta))^T$$

Let $N^i(\xi,\eta)$ denote the shape-function associated with the node i of an element and $\bar{n}$ the total number of nodes of the element.

Then $x_j = \sum_{i=1}^{\bar{n}} N^i(\xi,\eta) x_{ji} \qquad j = 1,2,3$

Where x_{ji} is the value of x_j at node i.

We also define the following tensor components.

$$E = \left|\frac{\partial \underset{\sim}{r}_p}{\partial \xi}\right|^2 \quad ; \quad F = \left| \frac{\partial \underset{\sim}{r}_p}{\partial \xi} \times \frac{\partial \underset{\sim}{r}_p}{\partial \eta} \right|$$

$$G = \left| \frac{\partial \underset{\sim}{r}_p}{\partial \eta} \right|^2 \quad ; \quad W = (EG-F^2)^{1/2}$$

The Beltrami first differential operator is given by :-

$$\underset{\sim}{\nabla}_\Gamma(u,u) = \left[G\left(\frac{\partial u}{\partial \xi}\right)^2 - 2F\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial u}{\partial \eta}\right) + E\left(\frac{\partial u}{\partial \eta}\right)^2\right] \Big/ W^2$$

5.4.1 Construction of Operators A_h and M_h :-

Let ϕ_h be the solution of the the discretized problem of (5.3), ie Given $u_h \in H_h^1(\Gamma_h)$ find $\phi_h \in V_h(\Omega_h)$ s. t.

$$\int_{\Omega_h} \underline{\nabla}\, \phi_h \cdot \underline{\nabla}\, v_h = \int_{\Gamma} u_h v_h \qquad (5.8)$$

$$\forall v_h \in V_h(\Omega_h)$$

Using standard finite element interpolations for ϕ_h and u_h we obtain the following discretized equations of (5.8) in matrix form.

$$\underline{\phi}_h = \underline{\underline{D}}\, \underline{u}_h$$

where $\underline{\phi}_h$ denotes the array of nodes of ϕ_h and $\underline{u}_h$ the array of nodes of u_h. Let $\underline{\phi}_\Gamma$ denote the nodes of $\underline{\phi}_h$ on Γ_h

Thus we have

$$\underline{\phi}_\Gamma = \underline{\underline{T}}_\Gamma\, \underline{\underline{D}}\, u_h$$

$$= \underline{\underline{H}}\, \underline{u}_h \qquad (5.9)$$

where $\underline{\phi}_\Gamma = \underline{\underline{T}}_\Gamma\, \underline{\phi}_h$ *and* $\underline{\underline{H}} = \underline{\underline{T}}_\Gamma \cdot \underline{\underline{D}}$

Here $\underline{\underline{H}}$ is termed as the coupling matrix.

We have the following weak form of (5.2) as the eigen-value problem

Find $(\lambda_h, u_h) \in \mathbb{R}^+ \times V_h(\Gamma_h)$ *s.t*

$$\int_{\Gamma_h} \underline{\nabla}_\Gamma (u_h, v_h) + \int_{\Gamma_h} bo u_h v_h + \int_{\partial\Gamma_h} \chi(u_h v_h = \lambda_h \int_{\Gamma_h} \phi_{\Gamma_h} v_h \qquad (5.10)$$

$$\forall\, v_h \in V_h\ (\Gamma_h)$$

and $\phi_{\Gamma_h} \in H_h^1\ (\Gamma_h)$

(5.10) can be expressed in the matrix form as the following :

$$\underline{v}_h^T \underline{\underline{A}}_h \underline{u}_h = \lambda_h \underline{v}_h^T \underline{\underline{M}} \phi_{\Gamma_h}$$

$$= \underline{v}_h^T (\lambda_h \underline{\underline{M}} \cdot \underline{\underline{H}} \underline{u}_h)$$

$$= \underline{v}_h^T \lambda_h \underline{\underline{B}}_h \underline{u}_h \text{ where } \underline{\underline{B}}_h = \underline{\underline{M}} \cdot \underline{\underline{H}}$$

$$= \underline{v}_h^T \lambda_h \underline{\underline{M}}_h \underline{u}_h \text{ where } \underline{\underline{M}}_h = \frac{1}{2}[\underline{\underline{B}}_h + \underline{\underline{B}}_h^T]$$

Hence the discretized eigen-value problem corresponding to (5.7) is the following :

Find $(\lambda_h, \underline{u}_h) \in \mathbb{R}^+ \times \mathbb{R}^{\bar{N}}$ s. t

$$\underline{\underline{A}}_h \underline{u}_h = \lambda_h \underline{\underline{M}}_h \underline{u}_h \tag{5.11}$$

Where $\bar{N}$ is the resultant dimension of the system.

(5.11) is a standard eigen-value problem which is also symmetric and positive definite and standard softwares are available for its solution. The operators $\underline{\underline{A}}_h$ and $\underline{\underline{M}}_h$ correspond to the stiffness and the Added-Mass operators under normal conditions.

5.5 NUMERICAL ALGORITHM

The major task in the numerical solution is the construction of the discretized operators $\underline{\underline{A}}_h$ and $\underline{\underline{M}}_h$. We shall now describe some details of the implementation of the method of their construction :

Step 1 The first task is to construct the free-surface. If it is not known a priori, the problem of the construction of the capillary surface was discussed in details in the

Chapters III and IV. Once the surface is constructed we perform the discretization of the domain and the free-surface. The surface need not be discretized separately. The domain is discretized using 3-dimensional (20-noded) solid elements and the sides of the element which lie on the surface of the liquid form the 2-dimensional discretization of the surface. The information regarding the coupling of the fluid nodes and the free-surface nodes are required for the construction of T_Γ matrix in (5.9).

Step 2 We now construct the $\underline{\underline{H}}$ matrix given in (5.9) by solving (5.8). Instead of directly inverting stiffness matrix in (5.8), we can store the $\underline{\underline{L}}\ \underline{\underline{D}}\ L^T$ factorization and solve for a load matrix which consists of the identity matrix to obtain the inverse. The advantage of this procedure is that some standard finite element method softwares can be used for static solution in a compacted storage form. The matrix H is stored in a secondary storage unit.

Step 3 The next step involves the computation of the matrices $\underline{\underline{A}}_h$ and $\underline{\underline{B}}_h$. They are computed at the element level. The matrix $\underline{\underline{A}}_h$ is similar to the ordinary stiffness matrix except the fact that some terms from the boundary also contribute to it. The matrices are computed on the basis of eq^n (5.8). It is to be noted that $\underline{\underline{M}}_h$ if fully populated and not diagonal which occur in the case of

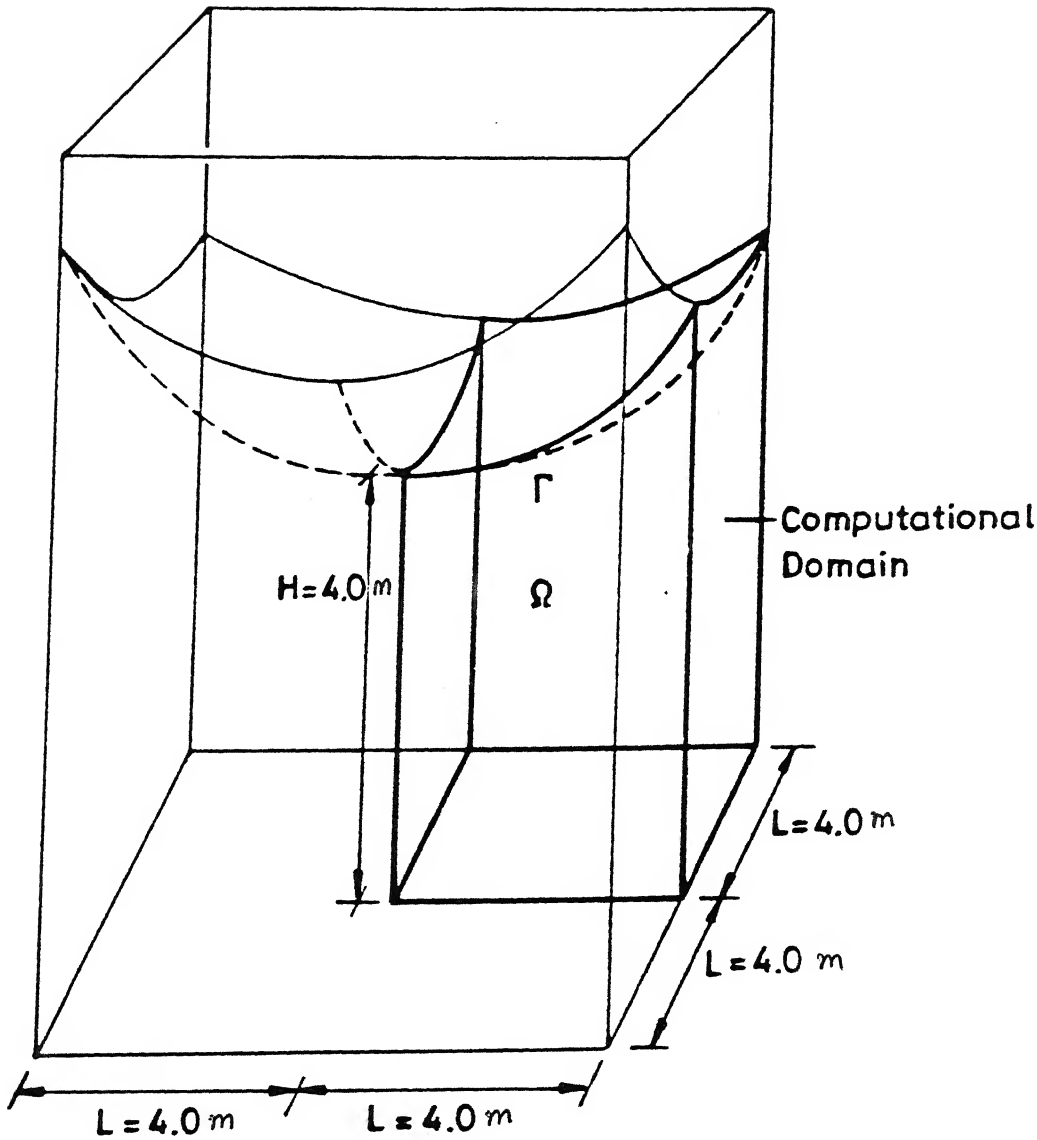
lumped mass matrix.

Step 4 After the matrices $\underline{\underline{A}}_h$ and $\underline{\underline{M}}_h$ are constructed, we directly invoke some eigen-solvers for positive definite matrices with compact storage mode. Since many finite-element softwares available for the dynamic run use the lumped mass matrix approach some significant changes are required to be made in the module to account for the fully populated mass matrix.

5.6 NUMERICAL RESULTS :

The proposed method is now applied to compute the fundamental eigen-values of the free-surface oscillations of an ideal liquid in a cylindrical container of square cross-section, under zero gravity conditions for various angles of contact (θ_c) (Fig. 5.3). From symmetry considerations only a quadrant is used for the computational purpose. The *3-D* mesh is depicted in Fig. 5.4.

For the domain occupied by the fluid *20-Noded solid element* [Zienkiewicz [1977]] and for the fluid surface *8-Noded Quadrilateral* are used. A *3-D* mesh generation routine has been developed for this purpose. The computed values of the fundamental eigen-values for various angles of contact is shown graphically in Fig 5.5. It may be noted that for the above geometry the free-surface does not exist for $\theta_c < 45^o$ [Finn [1986]]. For this minimum contact angle the eigen-value turns out to be negative. For higher contact angles positive eigen-values are obtained. The eigen-values show an increasing trend with an increase of the



System parameters :

Density of the liquid = 1000 kg / m^3

Surface Tension = 72.75 $\times 10^3$ Kg / s^2
(gas-liquid interface)

FIG. 5.3 Computational domain details.

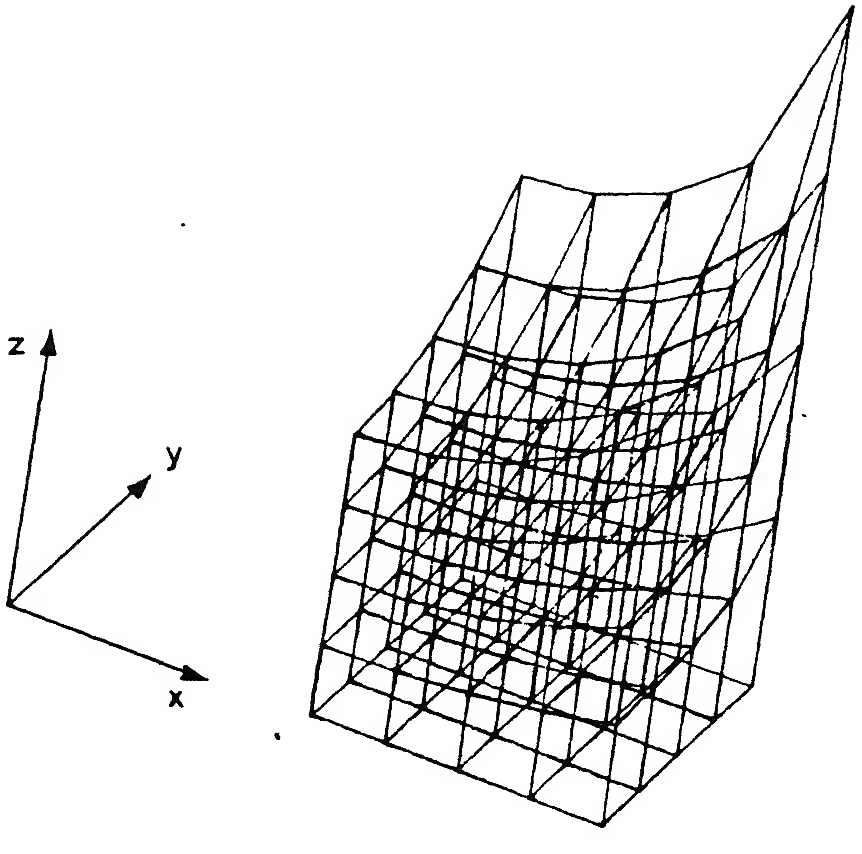


FIG. 5.4 3-D Finite Element Mesh.

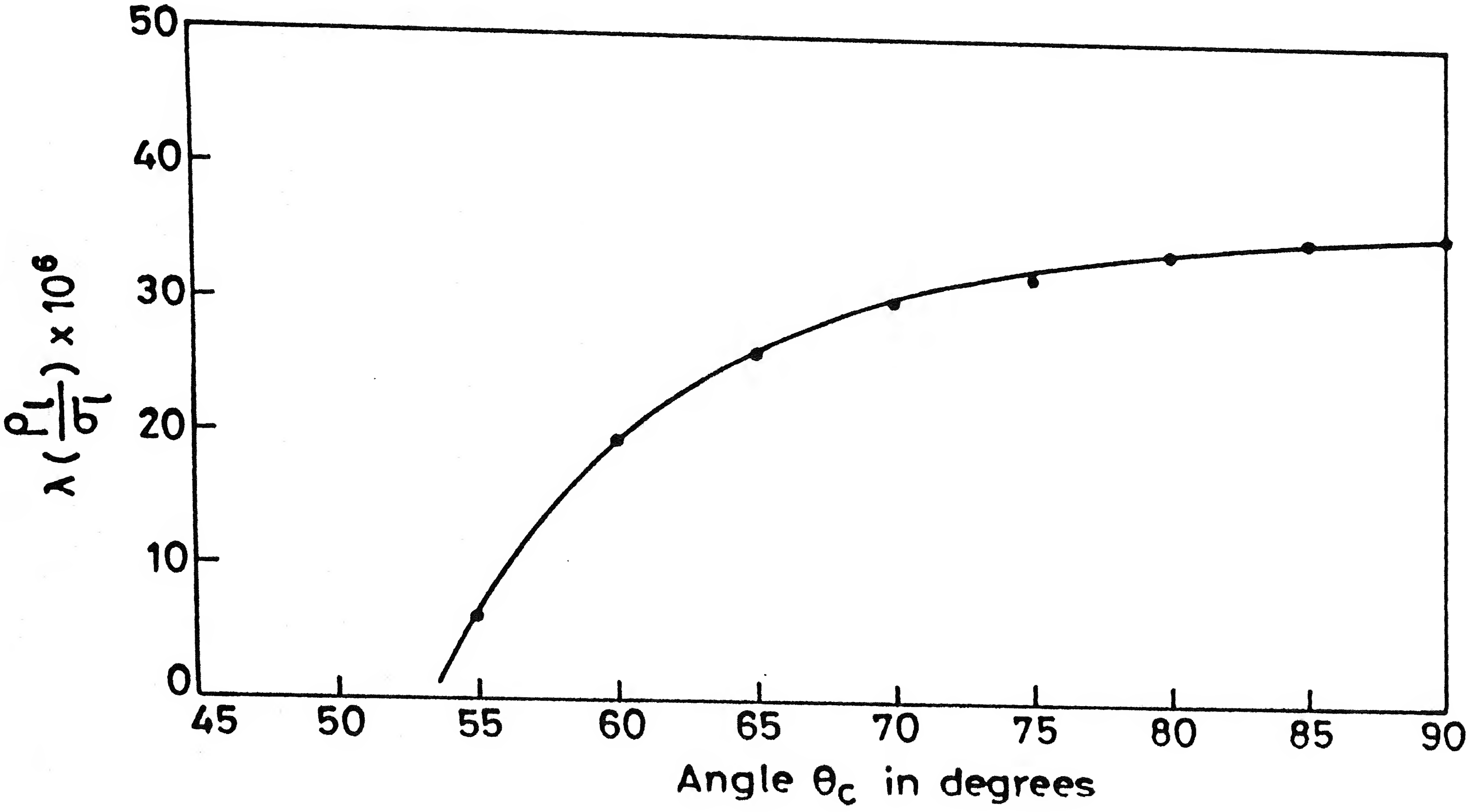


FIG.5.5 Plot of Eigen-values vs contact angle.

angles of contact up to 90^{o}. This trend is similar to the results published in [Bauer [1989c]] for the liquid oscillations in rectangular containers.

The results can be compared with the theoretical eigen-values of the oscillations of an ideal liquid in a cylindrical container of similar dimensions and with 90^{o} contact angle.

The fundamental eigen-value ($\tilde{\lambda}$) is 0.7E-05 units for the latter system. The order of magnitude of the eigen-values are almost same for the two systems which enhances the authenticity of the results.

5.7 DISCUSSIONS

With the construction of operators $\underline{\underline{A}}_h$ and $\underline{\underline{M}}_h$ thereby reducing the problem to a standard one a numerical method using the Finite Element Method has been successfully applied to solve the eigen-value problem. As one has to solve two different problems for the construction of operators the use of parallel processing becomes attractive. With this method the dimension of the problem is also greatly reduced. This is due to the operator $\underline{\underline{B}}_h$. One can also study the fluid-structure interaction by adding the structural stiffness and the mass operators with little difficulty.

CHAPTER VI

OSCILLATIONS OF LIQUIDS IN VESSELS WITH TUBE BUNDLES USING HOMOGENIZATION TECHNIQUE

6.1 INTRODUCTION :

In this chapter a method to analyse the oscillations of the an ideal liquid taking into the effect of capillary surface by applying the homogenization technique is presented. This is a study of the asymptotic behaviour of the eigen-values and eigen-vectors in a periodically perforated domain with the number of holes increasing to infinity. The distinguishing feature of this problem is that it is a 3-dimensinal one with a periodicity in 2-dimensions. The method of asymptotic expansion is used to obtain an explicit formula for the homogenized coefficients for the problem. Using the method of energy theorems on the estimates of the eigen value and their convergence are proved. Numerical examples are presented.

6.2 BACKGROUND AND LITERATURE SURVEY :

In the last few years some efforts have been made to study the vibrations of tube-bundles immersed in a fluid. Such problems arise in the mechanical behaviour of heat-exchangers, condensers, reactors etc. This problem has been numerically studied by Chen [1977], Paidoussis [1977], Schumann [1981], Shinohara [1981], Tanaka [1981], Planchard [1980,1982,1983], Planchard and Ibnou Zahir [1983] and many others.

In practice one is required to solve this kind of eigen-value problems where the number of tubes is very large [e.g. forty thousand in the core of a P.W.R]. If one uses finite element method, it is necessary to have large numbers of mesh points to represent the fluid between the tubes leading to matrices of large order (eg 10^5 for 2-D problems). The cases become worse for 3-D problems requiring still larger memory which may not be available on general computing machines and large C.P.U. time. Thus, there is a need to identify a simplified problem which is equivalent to the original problem. One such simplification is the limit-spectral problem which is a result of the asymptotic behaviour of the original problem. One such technique is the Homogenization technique.

Homogenization deals with the study of the limiting behaviour of the differential operators of physics in heterogeneous or perforated materials with a periodic structure of periodicity ϵ as $\epsilon \to 0$. For a heterogeneous material the material coefficients are not constant but vary with the space coordinates. In the case of a material with periodic structure the coefficients vary periodically. If the length of the period is very small one may think that the solution of the differential equation is approximately the same as the corresponding solution of a homogenized material with constant coefficients. The homogenization method, shows the existence of solution and derives some properties of the homogenized operator.

The theory of homogenization in perforated domains has been developed by many authors. The mathematical aspects of the

problem have been studied by Lions [1980, 1981]. Duvaut [1976], Cioranecu and Saint Jean Paulin [1979]. The eigen-value problem in perforated domains has been studied by Vanninathan [1978 a,b, 1981]. It is mainly the results of the works of Vanninathan which are useful to construct the homogenized operators in the problem of vibration of tube-bundles. Planchard [1980], Planchard and Ibnou-Zahir [1983] have studied the problem of vibration of tube-bundles using the theory of homogenization.

6.3 PROBLEM DESCRIPTION :

While considering the problem of the oscillations of the capillary surface of an ideal incompressible liquid the following assumptions are made :

1) The walls of the vessel are rigid and enclose a finite region in $\mathbb{R}^3$.
2) The tubes are rigid and are placed in a periodic manner.
3) The motion of the vessel takes place in low-gravity environment.

The present problem of tube-bundle vibrations is different from the problem studied by the earlier authors. The earlier studies neglected the effect of free-surface, which is justified under terrestrial conditions. Also the tubes were attached to flexible supports which provided the potential energy to the system while the liquid was considered to be incompressible. The problem also could be reduced to a 2-D one from the considerations of symmetry. In the environment of

low-gravity the effect of surface tension becomes pronounced and the study of the oscillations of the capillary surface becomes important. As the frequencies of oscillations of the capillary surface are much lower than the structural frequencies, the structure is considered to be rigid in the analysis. The elastic energy of the system is provided by the capillary surface which behaves like a stretched membrane and the kinetic energy is provided by the liquid motions. Thus the problem is a essentially 3-D one. This problem can be thought of as an extension of the study of the oscillations of the capillary surface in an ordinary vessel.

6.4 NOTATIONS AND PRELIMINARIES

Let us consider an open set $G \subset \mathbb{R}^3$ defined as follows :

$$G = \{(x_1, x_2, x_3) \in (o, l_1) \times (O, l_2) \times (o, l_3)\}$$

$(x_1, x_2, x_3)^T$ is a representation of a point in $\mathbb{R}^3$. l_1, l_2, l_3 are the dimensions of G along the 3-directions. G contains a large number of identical rigid tubes placed periodically in the x_1, x_2 directions. The axes of the tubes are parallel to x_3 [Fig. 6.1]

The boundary of G excluding the top surface is denoted by Σ_o and $\Omega = \partial G/\Sigma_o$ represents the top surface of the domain G. Σ_o represents the walls of the vessel.

We define the following set M

$$M = \{ (x_1, x_2, x_3) \in G;\ x_3 = \text{constant}\}$$

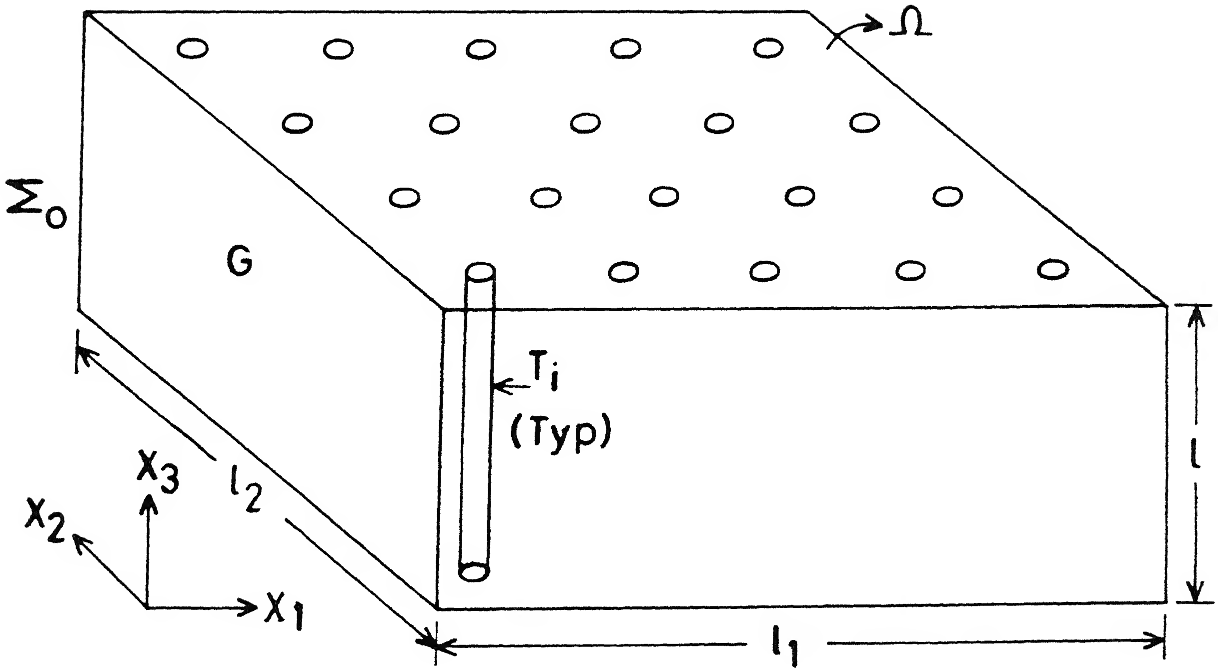


FIG. 6.1 A DESCRIPTION OF THE DOMAIN.

where M represents the cross section of domain (Fig 6.2). For the sake of convenience we define $\underline{x} = (x_1, x_2)^T$

Let $\underline{\nabla} = \left(\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \right)^T$

and $\Delta = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$

Let Y denote a representative cell [Fig 6.3] and T the hole. Any cross-section (normal to x_3) of the domain is composed of the translation of the cell Y along x_1 and x_2 directions.

We define the set $Y^* = Y/T$

Let $\epsilon > o$ denote a parameter associated with the periodicity of the system. The periodicity of the system will be $\epsilon\, y_1$ and $\epsilon\, y_2$ along the x_1 and x_2 directions respectively. This is also termed as $\epsilon\; Y$ periodicity.

We define $\theta = \frac{\text{meas}\,|Y^*|}{\text{meas}\,|Y|}$

The characteristic function $X(y)$ of Y is defined as following

$$X(\underline{y}) = 1 \;\forall\; \underline{y} \in Y^*$$

$$= 0 \;\forall\; \underline{y} \in T \cap Y$$

$X(\underline{y})$ is extended on $\mathbb{R}^2$ by periodicity.

The holes $T_j^\epsilon (j=1,2..)$ are defined as the closed connected components of the set

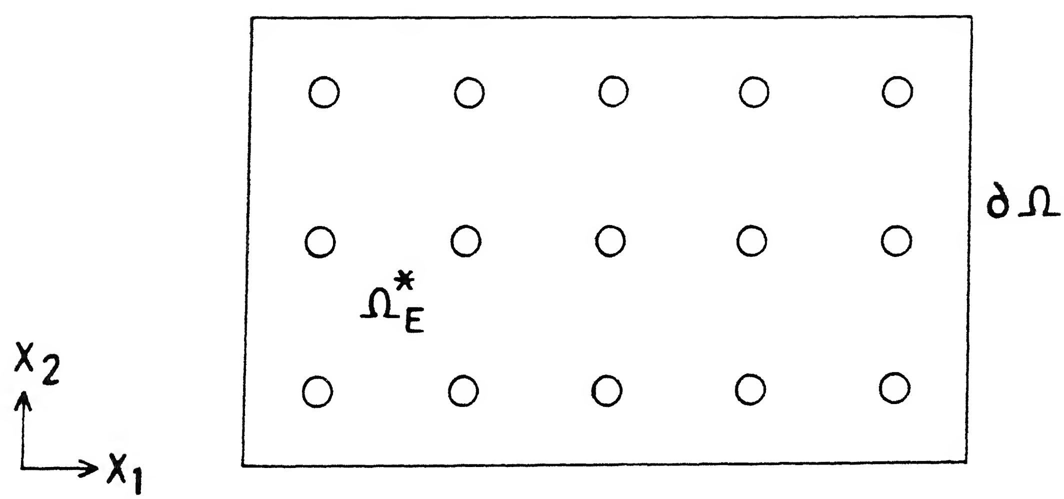


FIG. 6 2 A VIEW OF THE CROSS-SECTION OF THE DOMAIN.

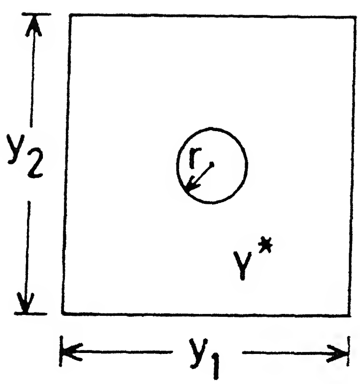


FIG. 6.3 DESCRIPTION OF A REPRESENTATIVE CELL Y.

$$\{\underline{x}:\ X(\underline{x}/\epsilon) = 0,\ \underline{x} \in \Omega\}$$

We shall denote $M_\epsilon^* = M \Big/ \bigcup_{j=1}^{N_\epsilon} {}_1T_j^\epsilon$

where N_ϵ denotes the number of tubes in the model.

Similarly $\Omega_\epsilon^* = \Omega \Big/ \bigcup_{j=1}^{N_\epsilon} {}_1T_j^\epsilon$

and $G_\epsilon^* = G \Big/ \bigcup_{j=1}^{N_\epsilon} {}_1T_j^\epsilon$

where in this case T_j^ϵ represents a tube correponding to the hole T_j^ϵ. We shall make the assumption that the holes do not intersect $\partial\Omega$. We denote by $m_Y(f)$ the average of f over Y.

$$m_Y(f) = \frac{1}{|Y|} \int_Y f(y)dy.$$

and $m_{Y*}(f)$ $\dfrac{1}{|Y^*|} \displaystyle\int_{Y*} f(y)dy$

Let us define the following function spaces :

$$V(\Omega_\epsilon^*) = \{v \in H^1(\Omega_\epsilon^*)\ ;\ \int_{\Omega_\epsilon^*} v = 0\}$$

$$V(\Omega) = \{v \in H^1(\Omega)\ ;\ \int_\Omega v = 0\}$$

$$V(G_\epsilon^*) = \{v \in H^1(G_\epsilon^*)\ ;\ \int_{G_\epsilon^*} v = 0\}$$

$$V(G) = \{v \in H^1(G) \ ; \int_G v = 0 \}$$

$$W = \{v \in H^1(Y^*) : v \text{ is } Y \text{ periodic}\}$$

6.5 MATHEMATICAL FORMULATION :

Let ϕ^ϵ denote the velocity potential of the fluid and u^ϵ the velocity of the capillary surface movement in the x_3 direction. Let $\underline{n}$ denote the outer normal to the domain under consideration. Since the walls of the vessel and the tubes are rigid the normal component of the fluid velocity on the walls and the tube surface is 0.

We denote the density of liquid by ρ, the surface-tension by σ and acceleration due to gravity by g. Let $bo = \rho g/\sigma$. We assume $bo \geq 0$ with $bo = 0$ when $g = 0$ (i.e., under zero gravity conditions) Let λ^ϵ denote the eigen-value of the system. In the liquid domain we have the following equations :

$$\left.\begin{aligned} - \Delta\phi^\epsilon + \frac{\partial^2 \phi^\epsilon}{\partial^2 x_3} &= 0 \quad \text{in } G^*_\epsilon \\ \frac{\partial \phi^\epsilon}{\partial \underline{n}} &= 0 \quad \text{on } \partial G^*_\epsilon / \Omega^*_\epsilon \\ \frac{\partial \phi^\epsilon}{\partial x_3} &= u^\epsilon \quad \text{on } \Omega^*_\epsilon \end{aligned}\right\} \qquad \text{(P-1)}$$

Along with the existence condition $\int_{\Omega^*_\epsilon} u^\epsilon = 0$

and the uniqueness condition $\int_{G^*_\epsilon} \phi^\epsilon = 0$

On the free-surface the following equations hold.

$$\left.\begin{aligned} -\Delta u^\epsilon + bou^\epsilon + c = \lambda^\epsilon \phi^\epsilon \qquad & in\ \Omega^*_\epsilon \\ \frac{\partial u^\epsilon}{\partial \underline{n}} = 0 \qquad & on\ \partial\Omega^*_\epsilon \end{aligned}\right\} \qquad \text{(P-2)}$$

Where c is a constant corresponding to the Lagrangian multiplier for the volume constraint. Since the admissible functions satisfy the constraint condition we can eliminate the constant c from the equation.

6.5.1 Variational Formulation :

In order to transform the problem (P-1)+(P-2) into a standard eigen-value problem in the weak form we construct the following operators.

We define $B^\epsilon : L^2(\Omega^*_\epsilon) \to L^2(\Omega^*_\epsilon)$ as follows. For a given $u^\epsilon \in V(\Omega^*_\epsilon)$ let ϕ^ϵ denote the solution of the following weak problem

Find $\phi^\epsilon \in V(G^*_\epsilon)$ s.t

$$\int_{G^*_\epsilon} \underline{\nabla}\, \phi^\epsilon . \underline{\nabla} v = \int_{\Omega^*_\epsilon} u^\epsilon v \quad \forall\ v \in V(G^*_\epsilon)$$

Then $B^{\epsilon}u^{\epsilon} = \phi^{\epsilon}\big|_{\Omega^*_{\epsilon}}$

We next define an operator A^{ϵ}: $H^1(\Omega^*_{\epsilon}) \to H^1(\Omega^*_{\epsilon})$ as follows :

$$(Au^{\epsilon}, v) = \int_{\Omega^*_{\epsilon}} \underline{\nabla}u^{\epsilon}.\underline{\nabla}v + \int_{\Omega^*_{\epsilon}} bo\; u^{\epsilon}v \qquad \forall u^{\epsilon}, v \in H^1(\Omega^*_{\epsilon})$$

We define M^{ϵ}_{a}: $[L^2(\Omega^*_{\epsilon})]$ as follows :

$$(M^{\epsilon}_{a}u^{\epsilon}, v) = (B^{\epsilon}u^{\epsilon}, v) = \frac{1}{2}\,[(B^{\epsilon}u^{\epsilon}, v) + (B^{\epsilon}v, u^{\epsilon})]$$

With the help of A^{ϵ}, B^{ϵ} and M^{ϵ} we obtain the following eigen-value problem of (P-1) + (P-2) in the standard weak form.

$$\left.\begin{aligned} &\text{Find } (u^{\epsilon}, \lambda^{\epsilon}) \in V(\Omega^*_{\epsilon}) \times \mathbb{R}^+ \text{ s.t} \\ &\quad (A^{\epsilon}u^{\epsilon}, v) = \lambda^{\epsilon}(B^{\epsilon}u^{\epsilon}, v) \\ &\qquad\qquad = \lambda^{\epsilon}(M^{\epsilon}_{a}u^{\epsilon}, v) \qquad \forall\, v \in V(\Omega^*_{\epsilon}) \end{aligned}\right\} \tag{P-3}$$

From the results of chapter V we note that $0 < \lambda^{\epsilon}_1 \le \lambda^{\epsilon}_2 \le \lambda^{\epsilon}_3 \le \ldots$ where λ^{ϵ}_j *is the* j^{th} eigen-value of (P-3).

We intend to study the following problem.

$$\left.\begin{aligned} &\text{Find } (u^h, \lambda^h) \in V(\Omega) \times \mathbb{R}^+ \text{ s.t} \\ &(A^h u^h, v) = \lambda^h (B^h u^h, v) \\ &\qquad\qquad = \lambda^h (M^h_a u^h, v) \qquad \forall\, v \in V(\Omega) \end{aligned}\right\} \tag{P-4}$$

where A^h, B^h and M_a^h are the homogenized operators obtained as an asymptotic limit as $\epsilon \to 0$ of the operators A^ϵ, B^ϵ and M_a^ϵ. (P-4) is termed as the homogenized problem of (P-3)

6.6 ASYMPTOTIC EXPANSION :

We now apply the method of asymptotic expansion to determine the homogenized coefficients for the problem. The method of asymptotic expansion was applied for the first time for homogenization problems in two dimensions for the perforated domains by Lions [1980, 1981]. We adopt a similar method for the 3-D eigen-value problem.

The asymptotic expansions of u^ϵ, ϕ^ϵ and λ^ϵ are the following :

$$u^\epsilon = u_o(\underline{x},\underline{y}) + \epsilon\, u_1(\underline{x},\underline{y}) + \epsilon^2 u_2(\underline{x},\underline{y}) + \tag{6.1}$$

$$\phi^\epsilon = \phi_o(\underline{x},x_3,\underline{y}) + \epsilon\, \phi_1(\underline{x},x_3,\underline{y}) + \epsilon^2\phi_2(\underline{x},x_3,\underline{y})+\ldots \tag{6.2}$$

$$\lambda^\epsilon = \lambda_o + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \ldots \tag{6.3}$$

We have $\Delta = \epsilon^{-2}\Delta_y + 2\,\epsilon^{-1}\Delta_{xy} + \Delta_x$

Also we have $\dfrac{\partial}{\partial \underline{n}} = \dfrac{1}{\epsilon}\,\dfrac{\partial}{\partial \underline{n}_y} + n_j(\underline{y})\,\dfrac{\partial}{\partial x_j}$ $\quad [j=1,2]$

Here Δ_x denotes the Laplacian w.r.t $\underline{x}$ variable and $\underline{n}_y$ is the normal at point $\underline{y}$.

From (P-2) we have

$$-\Delta u^\epsilon + bou^\epsilon = \lambda^\epsilon\phi^\epsilon \quad \text{in} \quad \Omega_\epsilon^* \tag{6.4}$$

Now substituting $u^{\epsilon}, \phi^{\epsilon}$ and λ^{ϵ} from (6.1), (6.2) and (6.3) in (6.4) and equating the 0 or -ve powers of ϵ to 0 we get

$$-\Delta_y u_o = 0 \tag{6.5}$$

$$-\Delta_y u_1 - 2\Delta_{xy} u_o = 0 \tag{6.6}$$

$$-\Delta_y u_2 - 2\Delta_{xy} u_1 - \Delta_x u_o + b_o u_o = \lambda_o \phi_o \tag{6.7}$$

From the boundary conditions on T^{ϵ} we get

$$\frac{\partial u_o}{\partial \underline{n}_y} = 0 \tag{6.8}$$

$$\frac{\partial u_1}{\partial \underline{n}_y} + n_j \left(\frac{\partial u_o}{\partial x_j} \right) = 0 \tag{6.9}$$

$$\frac{\partial u_2}{\partial \underline{n}_y} + n_j \left(\frac{\partial u_1}{\partial x_j} \right) = 0 \tag{6.10}$$

From (6.5) and (6.8) we obtain

$$u_o(\underline{x}, \underline{y}) = u_o(\underline{x})$$

From (6) we obtain

$$\Delta_y u_1 = 0 \qquad in\ Y^*$$

We introduce a function $X^j(\underline{y})$ as the following :

$$\left.\begin{aligned} -\Delta_y X^j &= 0 \qquad in\ Y^* \\ \frac{\partial X^j}{\partial n_y} &= n_j \qquad on\ T \end{aligned}\right\} \tag{P-5}$$

X^j is Y periodic, $j = 1,2$ and n_j the j^{th} component of the outward normals to T. It is easy to verify that

$$u_1(\underline{x},\underline{y}) = -X^j(\underline{y}) \left(\frac{\partial u_o}{\partial x_j} \right) + \tilde{u}_1(\underline{x}) \tag{6.11}$$

From (6.10) and (6.11) we obtain

$$\frac{\partial u_2}{\partial \underline{n}_y} = n_i \; X^j \left[\frac{\partial^2 u_o}{\partial x_i \partial x_j} \right] - n_i \frac{\partial \tilde{u}_1}{\partial x_i} \quad \text{on} \quad T \tag{6.12}$$

Thus from (6.7) and (6.10) we obtain

$$b_o u_o - \Delta_y u_2 + 2 \frac{\partial X^j}{\partial y_i} \left(\frac{\partial^2 u_o}{\partial x_i \partial x_j} \right) - \nabla_x u_o = \lambda_0 \phi_o \tag{6.13}$$

From the existence condition of solution of (6.13) we have

$$\int_{Y^*} b_o u_o - \int_T \frac{\partial u_2}{\partial n_y} + 2 \int_{Y^*} \frac{\partial X^j}{\partial y_i} \left[\frac{\partial^2 u_o}{\partial x_i \partial x_j} \right] - |Y^*| \nabla_x u_o = |Y^*| \lambda_o \phi_o \tag{6.13a}$$

Since $\int_T n_i(\underline{y}) = 0$ we have from (6.12) the following

$$\int_T \frac{\partial u_2}{\partial n_y} = \int_T n_i X^j \left[\frac{\partial^2 u_o}{\partial x_i \partial x_j} \right]$$

Let us denote by $\underline{v}$ a vector whichhas the i^{th} component unity and other components 0.

Then from (P-5) we obtain

$$\int_{Y^*} \underline{\nabla} X^j . \underline{v} = \int_T X^j \; \underline{v} . \underline{n}$$

[where we have used the relation $\underline{\nabla} X^j . \underline{\upsilon} = \underline{\nabla} . (X^j \underline{\upsilon}) - X^j \ \underline{\nabla} . \underline{\upsilon}$ and then the divergence theorem while noticing the fact that $\underline{\nabla} . \underline{\upsilon} = 0$ since $\underline{\upsilon}$ is constant]

Therefore

$$\int_{Y^*} \frac{\partial X^j}{\partial y_i} = \int_T n_i X^j$$

$$\therefore \int_T \frac{\partial u_2}{\partial \underline{n}_y} = \int_T \frac{\partial X^j}{\partial y_i} \left[\frac{\partial^2 u_0}{\partial x_i \partial x_j} \right]$$

Substituting $\int_T \frac{\partial u_2}{\partial \underline{n}_y}$ by $\int_T \frac{\partial X^j}{\partial y_i} \left[\frac{\partial^2 u_0}{\partial x_i \partial x_j} \right]$ in (6.13a) we get

$$\Rightarrow m_{Y*}(b_o)\, u_o - \nabla_x u_o + \frac{1}{|Y^*|} \left(\int_{Y^*} \frac{\partial X^j}{\partial y_i} \right) \left[\frac{\partial^2 u_o}{\partial x_i \partial x_j} \right] = \lambda_o \phi_o$$

Hence we have the homogenized equation

$$-q_{ij} \frac{\partial^2 u_o}{\partial x_i \partial x_j} + m_{Y*}(b_o) u_o = \lambda_o \, \phi_o \ \text{in } \Omega \qquad (6.14)$$

$$\text{where } q_{ij} = \frac{1}{|Y^*|} \int_{Y^*} \underline{\nabla} (X^j - y_j) . \ \underline{\nabla} (X^i - y_i) \qquad (6.14a)$$

q_{ij} are the homogenized coefficients. In obtaining q_{ij} we have used the fact that

$$\int_{Y^*} \underline{\nabla}(X^j - y_j) . \underline{\nabla} X^i = \int_T X^i \underline{\nabla}(X^j - y_j) . \underline{n} = 0 \text{ since } \underline{\nabla} (X^j - y_j) . \underline{n} = 0 \text{ on } T$$

If we denote by $[q_{ij}]_{i,j=1,2}$ as the matrix of the homogenized

coefficients, then we find that $[q_{ij}]$ is positive definite. This is easily seen as follows.

The condition for $[q_{ij}]$ to be positive definite is

$$q_{12} \le \sqrt{q}_{11} \sqrt{q}_{22} \text{ [since } [q_{ij}] \text{ is symmetric]}$$

$$\text{But } q_{12} = \frac{1}{|Y^*|} \int_{Y^*} \underset{-}{\nabla} (\chi^1 - y_1) . \underset{-}{\nabla} (\chi^2 - y_2) \le (\frac{1}{|Y^*|} \int_{Y^*} \underset{-}{\nabla} (\chi^1 - y_1) |^2)^{1/2}$$

$$(\frac{1}{|Y^*|} \int_{Y^*} \underset{-}{\nabla} (\chi^2 - y_2) |^2)^{1/2}$$

[from Cauchy-Schwartz inecquality]

We now try to express ϕ_o in terms of u_o so that (6.13) reduces to a standard eigen-value problem.

Using the asymptotic expansion (6.2) in (P-1) restricted to M^ϵ we obtain

$$\epsilon^{-2} \Delta_y \phi^\epsilon + 2\epsilon^{-1} \Delta_{xy} \phi^\epsilon + \Delta_x \phi^\epsilon + \frac{\partial^2 \phi^\epsilon}{\partial x_3^2} = 0 \quad \text{in } Y^*$$

$$-\Delta_y \phi_1 (\underset{-}{x}, \underset{-}{y}, x_3) = 0 \tag{6.15}$$

$$-\Delta_x \phi_2 (\underset{-}{x}, \underset{-}{y}, x_3) - 2\Delta_{xy} \phi_1 (\underset{-}{x}, \underset{-}{y}, x_3) - \Delta \phi_o (\underset{-}{x}, x_3) - \frac{\partial^2 \phi_0}{\partial x_3^2} = 0 \tag{6.16}$$

From the boundary conditions we obtain

$$\frac{\partial \phi_1}{\partial n_{\underset{-}{y}}} (\underset{-}{x}, \underset{-}{y}, x_3) + n_j \frac{\partial u_o}{\partial x_j} (\underset{-}{x}, x_3) = 0 \qquad \text{on } T$$

$$\frac{\partial \phi_1}{\partial \underline{n}_y} + n_j \frac{\partial u_o}{\partial x_j}(\underline{x}, x_3) = 0 \qquad \text{on } T$$

$$\frac{\partial \phi_o}{\partial x_3} = 0 \qquad \text{on } \Omega$$

Using $X^j(\underline{y})(j=1,2)$ from (P-5) we obtain from (6.15)

$$\phi_1(\underline{x}, \underline{y}, x_3) = - X^j(\underline{y}) \left(\frac{\partial \phi_o}{\partial x_j}(\underline{x}, x_3)\right) + \tilde{\phi}_1(\underline{x}, x_3) \quad j = 1,2$$

From (6.16) we obtain

$$-\Delta_y \phi_2 - 2\Delta_{xy}\phi_1 - \Delta\phi_o - \frac{\partial^2 \phi_0}{\partial x_3^2} = 0 \tag{6.17}$$

From the existence condition of (6.17) we obtain

$$2 \int_{Y^*} \frac{\partial X^j}{\partial y_i} \left[\frac{\partial^2 \phi_o}{\partial x_i \partial x_j} \right] - |Y^*| \Delta\phi_o - \int_T \frac{\partial \phi_2}{\partial \underline{n}_y} - |Y^*| \frac{\partial^2 \phi_0}{\partial x_3^2} = 0 \quad i,j=1,2$$

$$\Rightarrow - \Delta\phi_o + \frac{1}{|Y^*|} \left(\int_{Y^*} \frac{\partial X^j}{\partial y_i}\right) \left(\frac{\partial^2 \phi_o}{\partial x_i \partial x_j}\right) - \frac{\partial^2 \phi_0}{\partial x_3^2} = 0$$

Hence we have the homogenized problem corresponding to (P-1)

$$\left.\begin{aligned} -q_{ij} \frac{\partial^2 \phi_o}{\partial x_i \partial x_j} - \frac{\partial^2 \phi_o}{\partial x_3^2} &= 0 \quad \text{in } G \\ \frac{\partial \phi_o}{\partial \underline{n}} &= 0 \quad \text{on } \Sigma_o \\ \frac{\partial \phi_o}{\partial x_3} &= u_o \quad \text{on } \Omega \end{aligned}\right\} \tag{P-6}$$

Where q_{ij} coefficients are the homogenized coefficients defined in (6.14)

We define the homogenized operator B^h in the following way

Let ϕ_o be a solution of (P-6)

Then $\phi_o|_\Omega = B^h u_o$

Hence we obtain the standard eigen-value problem

$$\left.\begin{array}{l} -q_{ij} \dfrac{\partial^2 \phi_o}{\partial x_i \partial x_j} + m_{Y^*}(bo)u_h = \lambda^h B^h u_h \\ i,j = 1,\ 2 \end{array}\right\} \qquad \text{(P-7)}$$

where the coefficients q_{ij} are given in (6.14a)

6.7 HOMOGENIZATION THEOREMS :

Let us define the extension operators $\tilde{P}$ and $\tilde{Q}$ as follows :

$\tilde{P} \in \mathfrak{L}(H^1(Y^*),\ H^1(Y))$ s.t

$$|\underline{\nabla}\tilde{P}v|_{[L^2(Y)]^2} \le C|\underline{\nabla}v|_{[L^2(Y^*)]^2} \quad \forall\ v \in H^1(Y^*)$$

where $C > O$ is a constant, $\mathfrak{L}$ is the space of Bounded Linear Transformations. From Cioranescu [1979] we know that such an extension exists.

$\tilde{Q} \in \mathfrak{L}\ (L^2(\Omega^*_\epsilon), L^2(\Omega))$ is defined by extending the function to O value in the holes.

It is to be noted that the problem (P-4) is similar to the Neumann

eigen-value problem, the only modification is due to the presence of operator B^{ϵ} in place of the identity operator. For this we study the asymptotic property of B^{ϵ} in the following lemma.

Lemma 6.1

Let $\tilde{P}u^{\epsilon} \longrightarrow u^{*}$ in $L^{2}(\Omega)$

We have $\tilde{Q}(B^{\epsilon}u^{\epsilon}) \xrightarrow[\epsilon\to 0]{} \theta(B^{h}u^{*})$ in $L^{2}(\Omega)$,

where $B^{h}u^{*} = \phi^{*}|_{\Omega^{*}}$ where ϕ^{*} is obtained from the solution of the following problem

$$\left.\begin{aligned} \sum_{ij=1,2} q_{ij} \frac{\partial}{\partial x_i}\frac{\partial \phi^{*}}{\partial x_i} + \frac{\partial^2 \phi^{*}}{\partial x_3} &= 0 \qquad \text{in} \quad G \\ \frac{\partial \phi^{*}}{\partial \underline{n}} &= 0 \qquad \text{on } \Sigma_o \\ \frac{\partial \phi^{*}}{\partial x_3} &= u^{*} \qquad \text{on } \Omega \end{aligned}\right\} \qquad \text{(P-8)}$$

where q_{ij} are the coefficients given in (6.14a) which are obtained by the asymptotic expansion.

Proof : Since $\tilde{P}u^{\epsilon}$ is uniformly bounded, from the definition of B^{ϵ} we find that $\tilde{Q}(B^{\epsilon}u^{\epsilon})$ is bounded in $L^{2}(\Omega)$, hence we can extract a weakly convergent subsequence. We have to show below that the weak limit turns out to be $\theta\phi^{*}$.

Define $\xi_i^{\epsilon} = \frac{\partial \phi^{\epsilon}}{\partial x_i}$ $i = 1,2,3$. In view of the uniform boundedness of u_{ϵ}, $\left|\frac{\partial \phi}{\partial x_i}\right|$ are uniformly L_2 bounded and hence $\exists$ a subsequence s.t $\tilde{Q}\xi^{\epsilon} \xrightarrow[\epsilon\to 0]{} \xi^{*}$ in $L^{2}(G)$

We consider the weak form of (P-1) as follows:-

Let $v \in D(G)$. Then we have

$$\int_{G^*_\epsilon} \underline{\nabla}\phi^\epsilon \cdot \underline{\nabla} v + \int_{G^*_\epsilon} \frac{\partial\phi^\epsilon}{\partial x_3}\frac{\partial v}{\partial x_3} = \int_{\Omega^*_\epsilon} u^\epsilon v$$

or $$\int_G \tilde{Q}\underline{\xi}^\epsilon \cdot \underline{\nabla} v + \int_G \tilde{Q}\xi^\epsilon_3 \frac{\partial v}{\partial x_3} = \int_{\Omega^*_\epsilon} u^\epsilon v \tag{6.18}$$

Passing to the limit we obtain

$$\int_G \underline{\xi}^* \cdot \underline{\nabla} v + \int_G \xi^*_3 \frac{\partial v}{\partial x_3} = \theta \int_\Omega u^* v$$

Integrating by parts we obtain

$$\underline{\nabla}\cdot\underline{\xi}^* + \frac{\partial \xi^*_3}{\partial x_3} = 0 \qquad \text{in } G \tag{6.19}$$

$$\text{with } \xi^*_3 = \theta u^* \qquad \text{on } \Omega \tag{6.19a}$$

(6.19a) can be verified as follows

$$\lim_{\epsilon\to 0} \int_{G^*_\epsilon} \frac{\partial\phi^\epsilon}{\partial x_3} = \lim_{\epsilon\to 0} \int_l \int_{M^*_\epsilon} \frac{\partial\phi^\epsilon}{\partial x_3} = \theta \int_l \int_M \frac{\partial\phi^*}{\partial x_3}$$

Hence $\xi^*_3 = \theta \dfrac{\partial\phi^*}{\partial x_3}$

We need to obtain relation between ξ^* and ϕ^* in order to obtain the homogenized problem. We define an auxilliary problem for this purpose. For each $\underline{k} \in \mathbb{R}^3$ we define w_ϵ as the following. Find $w_k \in W$ s.t

$$\left.\begin{array}{l} \int_{Y^*} \underline{\nabla} w_k \cdot \underline{\nabla} v = 0 \quad \forall\, v \in W \\ \\ \text{and } w_k - \underline{k}\cdot\underline{y} \in W \quad \forall \underline{y} \in Y \end{array}\right\} \qquad \text{(P-9)}$$

Let $\eta_k = \underline{\nabla} w_k$

Hence from (P-9) we obtain

$$\left.\begin{array}{l} \int_{Y} \tilde{Q}\eta_k \cdot \underline{\nabla} v = 0 \quad \forall\, v \in W \\ \\ \quad \tilde{P}w_k - \underline{k}\cdot\underline{y} \in W \end{array}\right\} \qquad \text{(P-10)}$$

Let us introduce the functions

$$\tilde{w}_\epsilon(\underline{x}) = \epsilon\,(\tilde{P}w_k)(\underline{x}/\epsilon)$$

$$\eta_{k\epsilon} = \eta_k(\underline{x}/\epsilon)$$

$$\tilde{Q}\eta_{k\epsilon} = \tilde{Q}\eta_k(\underline{x}/\epsilon)$$

Then from (P-10) we have

$$-\underline{\nabla}.(\tilde{Q}\eta_{k\epsilon}) = 0 \text{ in } M^*_\epsilon \qquad (6.20)$$

We can extract subsequences $\{\tilde{w}_\epsilon\}$ and $\{\tilde{Q}\eta_{k\epsilon}\}$ *s.t*

$$\tilde{w}_\epsilon \longrightarrow \tilde{w}^* \text{ in } H^1(M) \qquad (6.21)$$

$$\underline{\nabla}\tilde{w}_\epsilon \longrightarrow \underline{k} \text{ in } [L^2(M)]^2 \qquad (6.22)$$

$$\tilde{Q}\eta_{k\epsilon} \longrightarrow m_Y(\tilde{Q}\eta_k) \text{ in } [L^2(M)]^2 \qquad (6.23)$$

$$\text{and } \underline{\nabla}\tilde{w}^* = \underline{k} \qquad (6.24)$$

An application of Theorem A.II.1 [Appendix II] gives us (6.23).

To obtain (6.21),(6.22), and (6.24) we introduce a function $\psi(\underline{y})$ $\forall$ $\underline{y} \in Y$ as follows

$$\psi(\underline{y}) = w_k(\underline{y}) - \underline{k}.\underline{y} \qquad [\underline{y} = \underline{x}/\epsilon]$$

or $\qquad \epsilon\psi(\underline{x}/\epsilon) = \epsilon w_k(\underline{x}/\epsilon) - \underline{k}.\underline{x}$

or $\qquad \tilde{w}_\epsilon(\underline{x}) = \underline{k}.\underline{x} + \epsilon\psi(\underline{x}/\epsilon)$

since $\psi(\underline{y}) \in L^2(Y)$, $\psi(\underline{x}/\epsilon) \in L^2(\Omega)$

Therefore $\tilde{w}_\epsilon \longrightarrow \underline{k}.\underline{x}$ in $L^2(\Omega)$

$\rightarrow \underline{\nabla}\tilde{w}_\epsilon \longrightarrow \underline{k}$ in $L^2(\Omega)$

and $\underline{\nabla}\,\tilde{\tilde{w}}^* = \underline{k}$

Let $v \in D(G)$. Multiplying (6.18) by $v\tilde{w}_\epsilon$ and (6.20) by $v\tilde{P}\phi^\epsilon$ we obtain

$$\iint_{lM} \tilde{Q}\underline{\xi}^\epsilon.\underline{\nabla}v\tilde{w}_\epsilon + \iint_{lM} \tilde{Q}\underline{\xi}^\epsilon.\underline{\nabla}v\tilde{w}_\epsilon - \iint_{lM} \tilde{Q}\eta_{k\epsilon}.\underline{\nabla}v\tilde{P}\phi^\epsilon$$

$$-\iint_{lM} \tilde{Q}\eta_{k\epsilon}\, v.\underline{\nabla}(\tilde{P}\phi^\epsilon) + \iint_{lM} \tilde{Q}\xi^\epsilon \frac{\partial v}{\partial x_3}\tilde{w}_\epsilon = \iint_{lM} u^\epsilon v\tilde{w}_\epsilon$$

We note that $\displaystyle\iint_{lM}(\tilde{Q}\underline{\xi}^\epsilon\, v\underline{\nabla}\tilde{w}_\epsilon - \tilde{Q}\eta_{k\epsilon}v.\underline{\nabla}(\tilde{P}\phi^\epsilon)) = 0$

This is as a result of the definitions of the extension operators.

Passing to the limit as $\epsilon \rightarrow 0$ we obtain

$$\iint_{lM} \xi\,.\underline{\nabla}v\tilde{\tilde{w}}^* - \iint_{lM} m_y(\tilde{Q}\eta_k)\underline{\nabla}v\phi^* + \iint_{lM} \xi_3^* \frac{\partial v}{\partial x_3}\tilde{\tilde{w}}^* = \theta\int_\Omega v^* v\tilde{w}$$

Integrating by parts and using (6.19) we obtain

$$\iint_{lM}(\xi^*.\underline{k} - \underline{m}_y(\tilde{Q}\eta_k).\underline{\nabla}\phi^*)v = 0$$

Since this is true for any $v \in D(G)$ we have

$$\xi^{*} \cdot \underline{k} = m_Y(\tilde{Q}\eta_k) \cdot \underline{\nabla}\phi^{*}$$

Substituting ξ^{*} and ξ_3^{*} in (6.19) we obtain

$$\int_G (m_Y(\tilde{Q}\eta_k)\underline{\nabla}\phi^{*}) \cdot \underline{\nabla}v + \int_G \theta \frac{\partial \phi^{*}}{\partial x_3} \frac{\partial v}{\partial x_3} = \theta \int_\Omega u^{*} v$$

$$\rightarrow \frac{1}{\theta} m_Y(\tilde{Q}\eta_k)\underline{\Delta}\phi^{*} + \frac{\partial^2 \phi^{*}}{\partial x_3} = 0 \quad \text{in } G$$

$$\frac{\partial \phi^{*}}{\partial x_3} = 0 \qquad \text{on } \Sigma_o$$

$$\frac{\partial \phi^{*}}{\partial x_3} = u^{*} \qquad \text{on } \Omega$$

The homogenized coefficients are given by the matrix $\frac{1}{\theta} m_Y (\tilde{Q}\eta_{\underline{k}})$. Since $\underline{k}$ is also arbitrary we may choose $\underline{k} = (1,0)^T$ and $(0,1)^T$ to obtain the coefficients in (6.14a) We can now state and prove the theorem on the estimates of eigen-values and the final theorem on the homogenization problem and conclude the theoritical analysis. We shall now prove a lemma which shall be useful in proving the homogenization theorem.

Lemma 6.2 : Let μ be an eigen-value of the following problem

$$-q_{ij} \frac{\partial^2 u_o}{\partial x_i \partial x_j} + m_{Y*}(bo)u = \mu B^{h} u \text{ in } \Omega$$

$$i,j = 1,2$$

$$\left.\begin{array}{l}\text{Where } q_{ij} \text{ are the coefficient defined in (6.14a).}\\ \text{Along with the boundary conditions}\\ \qquad\qquad \dfrac{\partial u}{\partial \underset{\sim}{n}} = o \quad \text{on } \partial\Omega\end{array}\right\} \qquad \text{(P-11)}$$

Let us define w^{ϵ} as the solution of the following problem

$$\left.\begin{array}{l} -\Delta w^{\epsilon} + bo\, w^{\epsilon} = \mu B^{h} u \quad \text{in } \Omega^{*}_{\epsilon} \\[2ex] \qquad\qquad \dfrac{\partial w^{\epsilon}}{\partial \underset{\sim}{n}} = o \quad \text{on } \partial\Omega^{*}_{\epsilon} \end{array}\right\} \qquad \text{(P-12)}$$

[where $w^{\epsilon} \in V(\Omega^{*}_{\epsilon})$]

Then the following assertion holds true

$$\tilde{P} w^{\epsilon} \longrightarrow u \text{ in } H^{1}(\Omega). \tag{6.25}$$

Proof : *From* (P-12) we obtain

$$\|w^{\epsilon}\|_{1,\Omega^{*}_{\epsilon}} \leq \tilde{C}\, \|B^{h}\|\, \|u\|_{o,\Omega}$$

$$\Rightarrow \qquad \|w^{\epsilon}\|_{1,\Omega^{*}} \leq Co \tag{6.26}$$

where $Co > o$ is a constant independent of ϵ.

From the property of the extension operator $\tilde{P}$

we know that $|\underset{\sim}{\nabla} \tilde{P} w^{\epsilon}|_{o,\Omega} \leq C_{1} |\underset{\sim}{\nabla} w^{\epsilon}|_{o,\Omega^{*}_{\epsilon}}$

With the help of Poincare-Wirtinger inequality and (6.26) we obtain

$$|\tilde{P}w^{\epsilon}|_{o,\Omega} \le C_2 |\underline{\nabla} w^{\epsilon}|_{o,\Omega} \le C_3 |\underline{\nabla} w^{\epsilon}|_{o,\Omega^*_{\epsilon}} \le C_4 .$$

Hence $\| \tilde{P}w^{\epsilon} \|_{1,\Omega} \le C_5$

where $C_5 > o$ is independent of ϵ.

Hence we can extract a subsequence again denoted by ϵ s.t. $\tilde{P}w^{\epsilon} \rightharpoonup w^*$ on $H^1(\Omega)$

We have to show that w^* satisfies the following equation

$$\left.\begin{array}{ll} -q_{ij} \dfrac{\partial^2 w}{\partial x_i \partial x_j} + m_y(bo)w^* = \mu B^h u & \text{in } \Omega \\ \dfrac{\partial w^*}{\partial \underline{n}} = O & \text{on } \partial\Omega \end{array}\right\} \quad \text{(P-13)}$$

Then comparing (P-13) with (P-11) we clearly find that $w^* = u$ [from the uniqueness of the solution]

In such a case we shall have $\tilde{P}w^{\epsilon} \rightharpoonup u$ in $H^1(\Omega)$ and (6.25) is proved consequently.

Let $\xi_i^{\epsilon} = \dfrac{\partial w^{\epsilon}}{\partial x_i}$ $i = 1,2$

As usual $\tilde{Q}$ shall denote the extension operator which extends the function by O in the holes.

It is clear from (6.26) that $\|\tilde{Q}\underline{\xi}^{\epsilon}\|_{O,\Omega^*_{\epsilon}} \le C$ [for some C independent of ϵ]

Thus we can extract a subsequence s.t. $\tilde{Q}\underline{\xi}^{\epsilon} \rightharpoonup \xi^*$ in $[L^2(\Omega)]^2$

From (P-12) we have

$$-\underline{\nabla}.\tilde{Q}\underline{\xi}^{\epsilon} + \chi^{\epsilon} bo \tilde{P} w^{\epsilon} = u\ B^{h} u \chi^{\epsilon} \quad \text{in } \Omega \tag{6.27}$$

$$\underline{\xi}^{\epsilon}.\underline{n} = 0 \qquad \text{on } \partial\Omega$$

From (P-13) we obtain $\forall \eta \in D(\Omega)$

$$\int_{\Omega} \tilde{Q}\underline{\xi}^{\epsilon}.\underline{\nabla}\eta + \int_{\Omega} \chi^{\epsilon} bo\ \tilde{P} w^{\epsilon} \eta = \mu \int_{\Omega} \chi^{\epsilon} B^{h} u\eta$$

Passing to the limit we obtain

$$\int_{\Omega} \underline{\xi}^{\epsilon}.\underline{\nabla}\eta + m_{y}(bo) \int_{\Omega} w^{*}\eta = \mu\theta \int_{\Omega} B^{h} u\eta$$

$$\Rightarrow -\underline{\nabla}.\underline{\xi}^{*} + m_{y}(bo)\ w^{*} = \mu\ \theta B^{h} u \quad \text{in } \Omega \tag{6.28}$$

We now define an auxilliary problem as follows :

For each $\underline{k} \in \mathbb{R}^{2}$ we define $\theta_{k} \in W$ as the solution of the following weak problem

$$\left.\begin{array}{l} \displaystyle\int_{Y^{*}} \underline{\nabla}\theta_{k}.\underline{\nabla}v = 0 \qquad \forall v \in W \\ \\ \text{and} \quad (\theta_{k} - \underline{k}.\underline{y}) \in W \qquad \forall \underline{y} \in Y \end{array}\right\} \tag{P-14}$$

with a uniqueness condition $\int_{Y^{*}} \theta_{k} = 0$

Let $\eta_{k} = \underline{\nabla}\ \theta_{k}$. Thus from (P-14) we obtain

$$\int_Y \tilde{Q}\eta_k \cdot \underline{\nabla}v = 0$$

Let us introduce the follwing functions needed to extend the functions defined in the unit cell to whole of Ω.

$$\tilde{\theta}_\epsilon = \epsilon \tilde{P}\, \theta_k(\underline{x}/\epsilon) \qquad \forall \underline{x} \in \Omega$$

$$\eta_{k\epsilon}(\underline{x}) = \eta_k(\underline{x}/\epsilon)$$

$$\tilde{Q}\eta_{k\epsilon} = \tilde{Q}\eta_k(\underline{x}/\epsilon)$$

From (P-14) we have

$$-\underline{\nabla}.(\tilde{Q}\eta_{k\epsilon}) = 0 \quad \text{in } \Omega. \tag{6.29}$$

We can now extract subsequences $\{\tilde{\theta}_\epsilon\}$ and $\{\tilde{Q}\eta_{k\epsilon}\}$ s.t.

$$\tilde{\theta}_\epsilon \rightharpoonup \tilde{\theta}^* \text{ in } H^1(\Omega) \tag{6.30}$$

$$\underline{\nabla}\tilde{\theta}_\epsilon \rightharpoonup \underline{k} \text{ in } [L^2(\Omega)]^2 \tag{6.31}$$

$$\tilde{Q}\eta_{k\epsilon} \rightharpoonup m_Y(\tilde{Q}\eta_k) \text{ in } [L^2(\Omega)]^2 \tag{6.32}$$

$$\text{and } \underline{\nabla}\tilde{\theta}^* = \underline{k} \tag{6.33}$$

(6.31) is a direct consequence of Theorem A.II.1 [Appendix II]

To obtain (6.30), (6.32) and (6.33) we first define a function $\psi(\underline{y})$ as follows:

$$\psi(\underline{y}) = \theta_k(\underline{y}) - \underline{k}.\underline{y} \qquad [\underline{y} = \underline{x}/\epsilon]$$

or $$\epsilon\psi(\underline{x}/\epsilon = \epsilon\theta_k(\underline{x}/\epsilon) - \underline{k}.\underline{x}$$

or $$\tilde{\theta}_\epsilon(\underline{x}) = \underline{k}.\underline{x} + \epsilon\psi(\underline{x}/\epsilon)$$

since $\psi(\underline{y}) \in L^2(Y)$, $\psi(\underline{x}/\epsilon) \in L^2(\Omega)$

Thus $\tilde{g}_\epsilon \longrightarrow \underline{k}.\underline{x}$ *in* $L^2(\Omega)$

Consequently we obtain (6.31) and (6.33)

Let $v \in D(\Omega)$. Multiplying (6.26) by $v\ \tilde{g}_\epsilon$, (6.27) by $v\tilde{P}w^\epsilon$ and subtracting we get [after integrating by parts]

$$\int_\Omega \tilde{Q}\underline{\xi}^\epsilon . \underline{\nabla} v\ \tilde{g}_\epsilon + \int_\Omega \tilde{Q}\underline{\xi}^\epsilon . \underline{\nabla}\tilde{g}_\epsilon v + \int_\Omega \chi^\epsilon bo\ \tilde{P}w^\epsilon v\tilde{g}_\epsilon - \int_\Omega \tilde{Q}\underline{\eta}_{k\epsilon} . \underline{\nabla} v\tilde{P}w^\epsilon$$

$$-\int_\Omega \tilde{Q}\underline{\eta}_{k\epsilon} . \underline{\nabla}(\tilde{P}w^\epsilon)v = \mu \int_\Omega \chi^\epsilon B^h u\ v\tilde{g}_\epsilon \qquad (6.34)$$

From the definitions of the operators $\tilde{Q}$ and $\tilde{P}$ we get

$$\int_\Omega \tilde{Q}\underline{\xi}^\epsilon . \underline{\nabla}\tilde{g}_\epsilon v \ -\int_\Omega \tilde{Q}\underline{\eta}_{k\epsilon} . \underline{\nabla}(\tilde{P}w^\epsilon)v = 0$$

[In the holes $\tilde{Q}\underline{\xi}^\epsilon = \tilde{Q}\underline{\eta}_{k\epsilon} = 0$ and in the region $\underline{\nabla}\tilde{g}_\epsilon = \tilde{Q}\underline{\eta}_{k\epsilon}$ and $\tilde{Q}\underline{\xi}^\epsilon = \underline{\nabla}(\tilde{P}w^\epsilon)$]

Passing to the limit in (6.31) we obtain

$$\int_\Omega \underline{\xi}^* \underline{\nabla} v\tilde{g}^* - \int_\Omega m_y(\tilde{Q}\underline{\eta}_k) . \underline{\nabla} w^* v + \int_\Omega m_y(bo)\ w^* v\tilde{g}^* = \mu\ \theta \int_\Omega B^h u\ v\tilde{g}^*$$

Integrating by parts and using (6.28) we obtain

It is clear that the dimension of $S_l|_{\Omega^*_\epsilon} = l$ for $\epsilon > 0$. Here $S_l|_{\Omega^*_\epsilon}$ denotes the subspace of S_l whose functions are restricted to Ω^*_ϵ. From the Mini-Max principle of the eigen-values we obtain

$$\lambda_l^\epsilon = Min \left\{ \underset{v\in S_l}{Max} \quad \frac{(A^\epsilon v, v)}{(B^\epsilon v, v)} \right\}$$

We can express v in terms of $\{w_j\}$ as the following

$$v = \sum_{j=1}^{l} \alpha_j w_j \text{ where } \alpha_j \text{ is scalar}$$

$$\therefore (A^\epsilon v, v) = (A^\epsilon(\sum_{j=1}^{l} \alpha_j w_j), \sum_{j=1}^{l} \alpha_j w_j) \le \sum_{j=1}^{l} \alpha_j^2 (A w_j, w_j) \le \mu_l (B^h v, v)$$

Hence $\lambda_l^\epsilon \le \mu_l \underset{v\in S_l}{Max} \dfrac{(B^h v, v)}{(B^\epsilon v, v)}$

We now have to prove $\underset{v\in S_l}{Max} \dfrac{(B^h v, v)}{(B^\epsilon v, v)} \le C_l$ (6.35)

to complete the proof of the theorem.

We assume the contrary. Then we can extract a subsequence of ϵ (again denoted by ϵ) and $v^\epsilon \in S_l$ *s.t.*

$$(B^h v^\epsilon, v^\epsilon) = 1 \tag{6.36}$$

$$\text{and} \quad (B^\epsilon v^\epsilon, v^\epsilon) \to 0 \tag{6.37}$$

It is clear that v^ϵ is bounded in $H^1(\Omega)$ since S_l is finite dimensional. Thus from Rellich's Compactness Theorem $\exists$ a

subsequence of v^ϵ weakly convergent in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Let us denote the limit by $\tilde{v}$.

Thus from (6.36) we obtain $(B^h\tilde{v},\tilde{v}) = 1$ and from (6.37) using Lemma 6.1 we obtain $(B^h\tilde{v},\tilde{v}) = o$ which is a contradiction. This proves (6.35) and hence the lemma.

We shall now present the final results in the form of two homogenizaion theorems.

Theorem 6.2 :

For any positive integer $l \geq 1$ let λ_l^ϵ be the l^{th} eigen-value of the problem (P-3) and u_l^ϵ the corresponding eigen-vector. Then $\{\lambda_l^\epsilon\}$ converges to an eigen-value λ_l^h and $\exists$ a sequence of ϵ s.t.

$$\tilde{P}u_l^\epsilon \rightharpoonup u^h \text{ in } H^1(\Omega).$$

where u^h is a solution of (P-7) corresponding to λ^h

Proof : Let us denote $\xi_i^\epsilon = \dfrac{\partial u_l^\epsilon}{\partial x_i}$ in Ω_ϵ^* $i = 1,2$

As usual $\tilde{Q}$ shall denote the extension operator in which the function is extended to 0 in the holes.

From Theorem 6.1 we conclude that $\|u_l^\epsilon\|_{1,\Omega} \leq \tilde{C}_l$ [where $\tilde{C}_l > 0$ is a constant independent of ϵ].

Thus we can extract a sequence of ϵ s.t.

$$\tilde{Q}\xi_i^\epsilon \rightharpoonup \xi_i^* \text{ in } L^2(\Omega) \;\; i = 1,2$$
$$\tilde{P}u_l^\epsilon \rightharpoonup \text{ in } H^1(\Omega)$$
$$\lambda_l^\epsilon \longrightarrow \lambda_l^h \text{ [since from Theorem 1 } \lambda_l^\epsilon \leq C_l \text{]}$$

Now (P-3) can be cast in the following form

$$\int_{\Omega^*_\epsilon} \xi^* . \underline{\nabla}\eta \;+\; \int_{\Omega^*_\epsilon} bO\; u^\epsilon_l \eta \;=\; \lambda^\epsilon_l \int_{\Omega^*_\epsilon} (B^\epsilon u^\epsilon_l)\eta \qquad \forall \eta \in D(\Omega) \tag{6.38}$$

Integrating by parts we obtain the following operator equation

$$-\underline{\nabla}.\xi^\epsilon \;+\; bou^\epsilon_l = \lambda^\epsilon_l B^\epsilon u^\epsilon_l \quad \text{in } \Omega^*_\epsilon \tag{6.39}$$

$$\xi^\epsilon . \underline{n} \;=\; o \text{ on } \partial\Omega^*_\epsilon$$

Passing to the limit in (6.38) and invoking Lemma 6.1 and Theorem A.II.1 (Appendix II) we obtain the following

$$-\underline{\nabla}.\xi^\epsilon \;+\; m_Y(bo)u^h \;=\; \lambda^h \theta\; B^h u^h \quad \text{in } \Omega \tag{6.40}$$

$$\xi^* . \underline{n} \;=\; O \qquad \text{on } \partial\Omega$$

We now define an auxilliary problem as follows :

Determine $w_k \in w$ s.t. for any $\underline{k} \in \mathbb{R}^2$

$$\left.\begin{array}{ll} \int_Y \underline{\nabla} w_k . \underline{\nabla} v = O & \forall\, v \in w \\ w - \underline{k}.\underline{y} \in w & \forall\, \underline{y} \in Y \end{array}\right\} \tag{P-15}$$

Together with a uniqueness condition $\int_{Y^*} w_k = O$

Let $\eta_k = \underline{\nabla}\, w_k$. Then from (P-12) we have

$$-\underline{\nabla} \cdot \tilde{Q}\eta_k = 0 \quad \text{in } \Omega \tag{6.41}$$

Let us introduce the following functions needed to extend the function defined in the unit cell to the whole of Ω.

$$\tilde{w}_\epsilon = \epsilon \tilde{P}w_k (\underline{x}/\epsilon) \qquad \forall\ \underline{x} \in \Omega$$

$$\eta_{k\epsilon} = \eta_k (\underline{x}/\epsilon)$$

$$\tilde{Q}\eta_{k\epsilon} = \tilde{Q}\eta_k(\underline{x}/\epsilon)$$

From (P-12) we have $-\underline{\nabla}.\tilde{Q}\eta_{k\epsilon} = 0$ in Ω

Let us introduce a function $\psi(y)$ as follows

$$\psi(\underline{y}) = w_k(\underline{y}) - \underline{k}.\underline{y} \qquad \forall\ \underline{y} \in Y$$

or $\tilde{w}_\epsilon = \underline{k} \cdot \underline{x} + \epsilon\ \psi(\underline{x}/\epsilon)\ [\underline{y} = \underline{x}/\epsilon]$

Since $\psi(\underline{y}) \in L^2(Y)$, $\psi\ (\underline{x}/\epsilon) \in L^2\ (\Omega)$

There $\tilde{w}_\epsilon \longrightarrow \underline{k}.\underline{x}$ in $L^2(\Omega)$.

We can now extract subsequences $\{\tilde{w}_\epsilon\}$ *and* $\{\tilde{Q}\eta_{k\epsilon}\}$ s. t.

$$\tilde{w}_\epsilon \longrightarrow \tilde{w}^* \text{ in } H^1(\Omega) \tag{6.42}$$

$$\underline{\nabla}\tilde{w}_\epsilon \longrightarrow \underline{k} \text{ in } [L^2(\Omega)]^2 \tag{6.43}$$

$$\tilde{Q}\eta_{k\epsilon} \longrightarrow m_Y(\tilde{Q}\eta_k) \text{ in } [L^2(\Omega)]^2 \tag{6.44}$$

$$\text{and } \underline{\nabla}\tilde{w}^* = \underline{k} \tag{6.45}$$

Let $v \in D(\Omega)$. Multiplying (6.39) by $v\tilde{w}_\epsilon$ and (6.41) by $v\tilde{P}u_l^\epsilon$ and substracting we get

$$\int_\Omega \tilde{Q}\xi^\epsilon . \underline{\nabla} v \; \tilde{w}_\epsilon + \int_\Omega \tilde{Q}\xi^\epsilon . \underline{\nabla}\tilde{w}_\epsilon v + \int_\Omega \chi^\epsilon b_0 \; \tilde{P}u_l^\epsilon v \tilde{w}_\epsilon - \int_\Omega \tilde{Q}\eta_{k\epsilon} . \underline{\nabla} v \tilde{P} u_l^\epsilon$$

$$-\int_\Omega \tilde{Q}\eta_{k\epsilon} . \underline{\nabla}(\tilde{P}u_l^\epsilon) v = \lambda_l^\epsilon \int_\Omega \tilde{Q}(B^\epsilon u_l^\epsilon) v \tilde{w}_\epsilon \tag{6.46}$$

From the definitions of the operators $\tilde{P}$ and $\tilde{Q}$ we get

$$\int_\Omega [\tilde{Q}\xi^\epsilon . \underline{\nabla}\tilde{w}_\epsilon - \tilde{Q}\eta_{k\epsilon} . \underline{\nabla}(\tilde{P}w^\epsilon)] v = 0$$

[Since in the holes $\tilde{Q}\xi^\epsilon = \tilde{Q}\eta_{k\epsilon} = 0$ and in the region $\underline{\nabla}\tilde{w}_\epsilon = \tilde{Q}\eta_{k\epsilon}$ and $\tilde{Q}\xi^\epsilon = \underline{\nabla}(\tilde{P}w^\epsilon)$]

Passing to the limit in (6.46) and using (6.42), (6.43), (6.44), (6.45) we obtain

$$\int_\Omega (\xi^* . \underline{k} - m_y \langle \tilde{Q}\eta_k \rangle . \underline{\nabla} u^h) \; v = 0$$

Since $v \in D(\Omega)$ is arbitrary we have

$$\xi^* . \underline{k} = m_y \langle \tilde{Q}\eta_k \rangle . \underline{\nabla} \; u^h$$

Since $\underline{k}$ is also arbitrary we may choose $\underline{k} = (0,1)^T$ and $(0,1)^T$ to obtain the coefficients q_{ij} given in (6.14a). This proves the theorem.

Theorem 6.3 :

Let $\{\lambda_l^\epsilon\}$, $\{u_l^\epsilon\}$ be the sequences of eigen-values and the corresponding eigen-vectors of the problem (P-7). Then the following hold.

(i) $\lambda_l^\epsilon \underset{\epsilon \to o}{\longrightarrow} \lambda_l$

(ii) $\exists$ a sequence of $\{\epsilon_p\}$(again denoted by ϵ) s.t

$\tilde{P}u_l^\epsilon \longrightarrow u_l$ in $H^1(\Omega)$

Proof : – From Theorem 6.2, (ii) is clear.

Finally it remains to prove that the limiting point λ_l^h of $\{\lambda_l^\epsilon\}$ is the l^{th} eigen-value of (P-7). For this it suffices to verify that

(i) there is no eigen-value ~~other~~ than $\{\lambda_l^h\}_{l=1}^\infty$ for (P-7) (6.47)

(ii) $\{u_l^h\}_{l=1}^\infty$ is an orthogonal basis in $L^2(\Omega)$ – w.r.t the inner product $(B^h(.),.)$ (6.48)

For proving (6.47) we argue by contradiction.

Let us suppose that $\exists$ an eigen-value μ different from λ_l^h with eigen-vector u which satisfied the homogenized problem (P-7) i.e.

$$-q_{ij}\,\frac{\partial^2 u_o}{\partial x_i \partial x_j} + m_y*(bo)u = \mu\, B^h u \qquad \text{in } \Omega$$

$$i,j = 1,2$$

$$u \in V(\Omega)$$

$$\mu \neq \lambda_l^h \ \forall\, l$$

$$\int_\Omega (B^h u)\, u^h{}_l = o$$

and $\int_{\Omega} (B^h u) u^h_l = 1$

We can choose an integer l s.t. $\mu < \lambda^h_{l+1}$

We define w^ϵ as the solution of the following problem

$$-\Delta w^\epsilon + bo\, w^\epsilon = \mu\, B^h u \quad \text{in} \quad \Omega^*_\epsilon$$

$$\frac{\partial w^\epsilon}{\partial \underline{n}} = 0 \qquad \text{on } \partial\Omega^*_\epsilon$$

Then from Lemma 6.2 we have $\tilde{P} w^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ Now let us define the function v^ϵ as follows

$$v^\epsilon = w^\epsilon - \sum_{k=1}^{l} (B^\epsilon w^\epsilon, u^\epsilon_k) u^\epsilon_k \tag{6.49}$$

$$\therefore \int_{\Omega^*_\epsilon} B^\epsilon v^\epsilon u^\epsilon_j = \int_{\Omega^*_\epsilon} B^\epsilon w^\epsilon u\; u^\epsilon_j - \sum_{k=1}^{l} \int_{\Omega^*_\epsilon} (B^\epsilon w^\epsilon, u^\epsilon_k) B^\epsilon u^\epsilon_k u^\epsilon_j$$

$$= \int_{\Omega^*_\epsilon} B^\epsilon w^\epsilon u^\epsilon_j - \int_{\Omega^*_\epsilon} B^\epsilon w^\epsilon u^\epsilon_j = 0 \text{ for } j = 1, 2, \ldots l$$

Since $\lambda^\epsilon_{l+1} = \inf\limits_{v^\epsilon \in (\Omega^*_\epsilon)} \dfrac{A^\epsilon(v^\epsilon, v^\epsilon)}{(B^\epsilon v^\epsilon, v^\epsilon)}$

$$A^{\epsilon}(v^{\epsilon},v^{\epsilon}) \quad \geq \lambda^{\epsilon}_{l+1} \ (B^{\epsilon}v^{\epsilon},v^{\epsilon}) \tag{6.50}$$

$$\lim_{\epsilon \to 0} \int_{\Omega^*_\epsilon} B^{\epsilon}w^{\epsilon}u^{\epsilon}_k = \lim_{\epsilon\to 0} \int_{\Omega} \tilde{Q}(B^h w^{\epsilon})u^{\epsilon}_k = \theta \int_{\Omega} B^h u \ u^h_k = 0$$

Therefore $\tilde{P}v^{\epsilon} \rightharpoonup u$ in $H^1(\Omega)$

$$\therefore A^{\epsilon}(v^{\epsilon},v^{\epsilon}) = \int_{\Omega^*_\epsilon} \underline{\nabla} (\tilde{P}v^{\epsilon}) . \ \underline{\nabla}(\tilde{P}v^{\epsilon}) + \int_{\Omega^*_\epsilon} bo(\tilde{P}v^{\epsilon})^2$$

Hence $\lim_{\epsilon\to 0} A^{\epsilon}(v^{\epsilon},v^{\epsilon}) = \int_{\Omega} \underline{\nabla}u . \underline{\nabla}u + m_Y \ (bo) \ u = \mu\theta \int_{\Omega} (B^h w)u = \mu\theta$

Again $\lim_{\epsilon\to 0} \int_{\Omega^*_\epsilon} \tilde{Q}(B^h v^{\epsilon}) \ \tilde{P}v^{\epsilon} = \theta \int_{\Omega} (B^h u^{\epsilon})u \ = \theta$

Hence passing to the limit in (6.50) we obtain

$\mu\theta \geq \lambda^h_{l+1} \ \theta$

or $\mu \geq \lambda^h_{l+1}$ which is a contradiction.

Thus (6.47) is proved.

To prove (6.48) we proceed in a similar fashion. Define a function $v \in V(\Omega)$ as follows.

$$v = w - \sum_{j=1}^{l} ((B^h w),u^h_j \ )u^h_j \qquad j = 1,2 \ldots l$$

for any $w \in V(\Omega)$

Thus we obtain $A^h(v,v) \geq \lambda_{l+1}^h \; (B^h v,v)$

Since $A^h(v,v) \leq C$ [where $C > 0$ is some constant]

We have $\lim_{l \to \infty} (B^h v,v) \longrightarrow 0$

$\to v = 0$. This proves that $\{u_l^h\}$ form a complete orthogonal basis in $L^2(\Omega)$ w.r.t. the inner product $(B^h(.),.)$.

6.9 NUMERICAL RESULTS :

We have computed the eigen-frequencies of system depicted in Fig.6.1 for various θ [where θ is the ratio of the dimension of the tube to the dimension of the unit cell]. The dimensions of the vessel are 4 x 4 x 4 m and the liquid considered is water.

The finite-element mesh is shown in Fig. 6.5. The elements used are the 20 noded 3-D solid elements for the liquid and 8-noded isoparametric elements for the free-surface discretization.

The discretization of the representative cell for the purpose of computing the homogenization coefficients is given in Fig.6.4. The variation of the first five frequencies with the θ is given in Table 6.1. Five different ratios have been considered and for a given θ the five eigen values are listed in the table.

6.10 DISCUSSIONS :

A method has been developed for the computation of the eigen-values of the oscillations of an ideal incompressible liquid

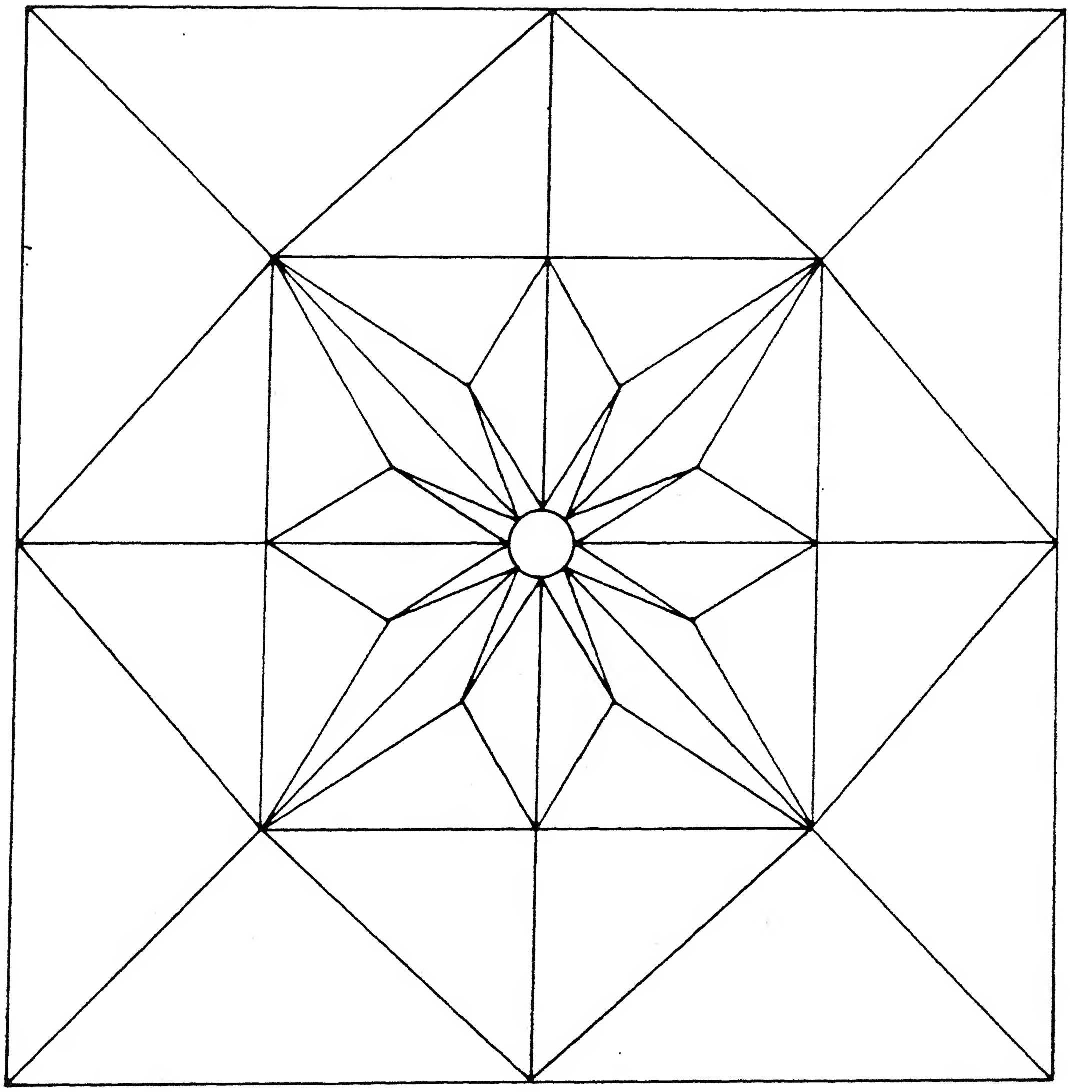


FIG. 6.4 2-D MESH OF THE REPRESENTATIVE CELL.

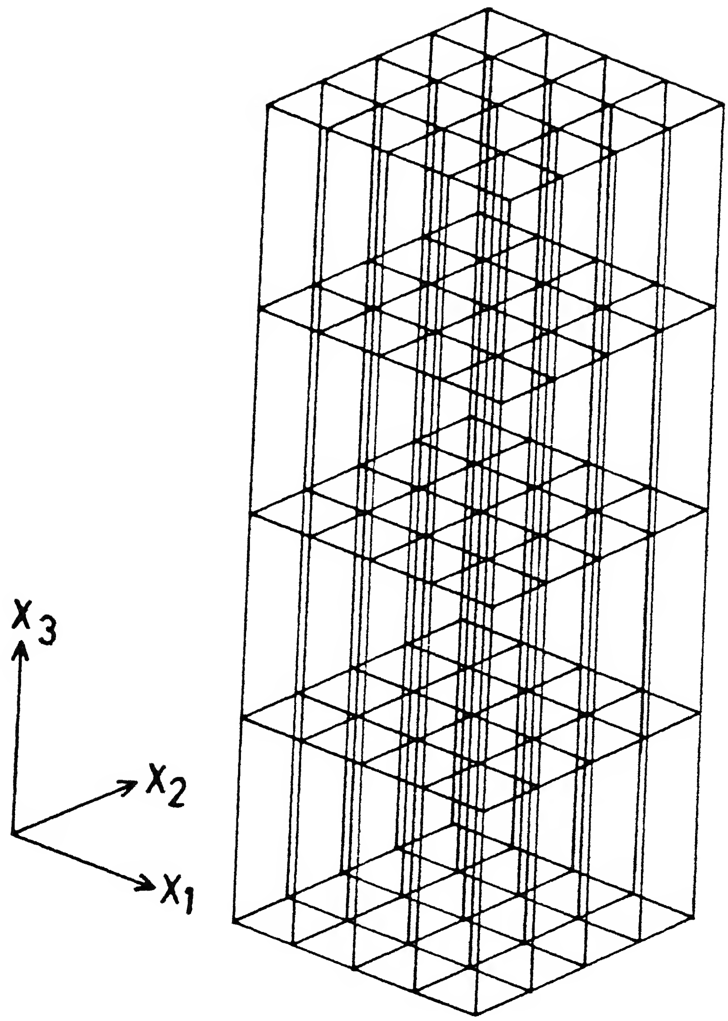


FIG. 6.5 3-D MESH

Table 6.1

Table showing the first five eigen-values for various θ.

	θ				
	0.75	0.5	0.25	0.125	0.0
$\lambda\ (\frac{\rho_l}{\sigma_l})$	$.1649\times10^{-3}$	$.6361\times10^{-4}$	$.3724\times10^{-4}$	$.3486\times10^{-4}$	$.3515\times10^{-4}$
	$.1649\times10^{-3}$	$.6361\times10^{-4}$	$.3724\times10^{-4}$	$.3486\times10^{-4}$	$.3515\times10^{-4}$
	$.4689\times10^{-3}$	$.1803\times10^{-3}$	$.1057\times10^{-3}$	$.9904\times10^{-4}$	$.9984\times10^{-4}$
	$.1369\times10^{-3}$	$.5200\times10^{-3}$	$.3038\times10^{-3}$	$.2845\times10^{-3}$	$.2868\times10^{-3}$
	$.1369\times10^{-3}$	$.5200\times10^{-3}$	$.3038\times10^{-3}$	$.2845\times10^{-3}$	$.2868\times10^{-3}$

using homogenization techniques. It can be seen that since the domain is composed of the translation of the representative cell, a single element computation [for stiffness or mass matrices] can be used for the global assembly. This brings about a significant reduction in the C.P.U. time and the computer memory.

From the numerical results it is observed that the frequencies tend to the frequencies of the system comprising of the liquid only and without the tubes as $\theta \rightarrow 0$. This is because the homogenized coefficients $q_{ii} \rightarrow 1$ $q_{ij} \rightarrow 0$ as $\theta \rightarrow 0$. This means that the effect of the tube becomes lesser as the ratio θ decreases. Also on the other hand when $\theta \rightarrow 1$ we also do not expect the results to be of much significance since this means that the tubes touch each other and the continuity of the region is lost.

CHAPTER VII

CONCLUSIONS

7.1 CONCLUSIONS ON THE RESULTS ACHIEVED :-

The first problem which has been considered is the construction of the capillary surface which in its generality is a free-boundary problem. For its solution a shape-optimization technique has been developed and applied to some problems. In this method the capillary surface problem is reformulated as a minimization problem in which the modified energy functional [which accounts for the volume constraint] is minimized with respect to the domains and a proof for the existence of the solution has been given. The major advantage of the method is that an accurate and inexpensive technique has been developed to compute the gradient of the non-linear energy functional. Hence it has become possible to use an efficient optimization algorithm based on the gradient for the numerical solution. In the present work the 3-noded triangles have been used for the finite-element formulation. However it is possible have more accurate representation of the surface using higher order isoparametric elements.

The prime task of an adaptive refinement scheme is to develop an efficient error indicator. Although the operator occurring in the capillary surface problem is strongly monotone, the error indicator based on the residual error estimate, does not perform satisfactorily due to the fact that the error tends to get normalized and regions of high gradients are not detected easily. In the work an adaptive refinement scheme based on a new error

indicator has been developed which performs satisfactorily. From the computational point of view, the efficiency of the method will depend on the data structure for the finite element mesh and its refining and the method used.

The problems considered in the work consisted of simple geometries [square section] using simple mesh generation methods. However, for complex geometries one has to explore other mesh generation techniques in order to obtain an efficient one. For the non-linear solution the full Newton's Method which involves the computation of Jacobian, has been used. Since the Jacobian is computed at the element level, the finite-element method is ideally suited for the Newton's Method, although some choice of the relaxation parameter can be made from efficiency considerations. For the large systems one can use either the sky-line solver with block storage or the frontal solvers. Although in the work the sky-line solver with block storage scheme has been used it may be explored to use frontal solvers in order to reduce the C.P.U. time.

The variational formulation for the problem of the linear oscillations of the free-surface becomes difficult due to the fact that the differential operator appearing on the boundary is of higher order than the differential operator appearing in the domain. This problem is re-formulated into a standard eigen-value problem with the help of some operators resembling the normal stiffness and mass operators. The difficulty in the computation arises in the finite-element modelling of the surface. Inacurrate modelling of the surface may give rise to large discretization

errors, because they occur throughout the domain and not only near the boundary as in the case of planar 2-dimensional problems. For the eigen-solution several finite-element softwares are available which are based on the subspace iteration method or the Lanczos' method. The significant modification which has to be carried out in this work is in the assembly of the mass matrix which is fully populated unlike the lumped mass matrix.

The problem of the computation of the eigen-values for the liquid oscillations containing large number of rigid bundles poses computational difficulties due to the requirement of large number of finite-element nodes for its solution. It has been solved using the method of homogenization which involves the construction of the homogenized operators for the eigen-value problem. This is basically a 3-dimensional problem with a periodicity in 2-dimensions and requires the construction of the homogenized operators both in the liquid domain as well as at the free-surface. The convergence property of the eigen-values and the eigen-vectors have been studied in the form of homogenization theorems.

7.2 SCOPE OF FURTHER WORK

There is a lot of scope of further work in the numerical aspects of the capillary surface problem. In the work the numerical scheme for the determination of the capillary surface

was presented for the case in which it is representable as the graph of function. The shape-optimization technique so presented can be also applied to the generalized 3-dimensional problem. Of course in that situation the algorithm will become more complex mainly due to the 3-dimensional node movements which requires efficient data structures and adequate graphics-support.

For the dynamic problems, especially the transient problems, the number of results is also very small. The theory of the dynamic angle of contact is a much debated one and there is not agreement on the question whether the dynamic contact angle shows a hysteresis phenomenon.

The finite-element adaptive strategy has been developed for the capillary surface problems without singularity. Now the method can be extended to study some problems with singularity. There is also a good scope of developing a finite-element based adaptive strategy for the free-surface flows with surface tension. This seems to be a complex task as the surface tension also appears in the problem along with the Navier-Stokes equation

The finite-element method which has been used for the computations of the eigen-frequencies of oscillations of an ideal liquid can be used to study the oscillations of two imiscible fluids. It can also be used for studying forced oscillations and also the fluid-structure interaction problem taking all phenomena

into account viscosity and also behaviour can be studied in sufficient details .

APPENDIX I

A.I.1 FORMULATION OF THE PROBLEM OF LINEAR OSCILLATIONS OF AN IDEAL LIQUID .

In this section we develop the equations governing the free-surface oscillations of an ideal liquid under low gravity conditions. The vessel which contains the liquid is assumed to be rigid and the liquid to be ideal and incompressible. We also assume that the equilibrium configuration of the capillary surface to be known a priori. In the sequel we shall follow the notations introduced in Chapter V.

In the liquid region we have the potential flow equation

$$\Delta\phi = 0 \quad \text{in } \Omega \tag{1}$$

As regards the boundary condition we have the no-flow condition on the wetted part of the vessel surface.

$$\frac{\partial\phi}{\partial \underline{n}} = 0 \quad \text{on } \Sigma \tag{2}$$

As a kinematic condition we have

$$\frac{\partial\phi}{\partial \underline{n}} = u \quad \text{on } \Gamma \tag{3}$$

where $u(\underline{x})$ is the velocity of a point [with coordinates $\underline{x}$] on Γ. This means that the velocity of the liquid on the free-surface matches with the velocity with which the free-surface is displaced along the normal.

Also since the volume of the liquid is conserved we have the condition $\int_\Gamma u = 0$ or $\int_\Gamma \frac{\partial \phi}{\partial n} = 0$ (4)

A. I.2 Dynamic Conditions on the Equilibrium Surface .

In equation (1-4) the capillary properties (i.e. surface-tension (denoted by σ), angle of contact, the curvatures of the surface) were not required. These properties are manifested in the dynamic conditions on Γ. We shall apply Hamilton's principle to derive the condition. In the system under consideration where the structure is considered to be rigid the kinetic energy is provided by the liquid mass and the potential energy by the free-surface which behaves like a stretched membrane.

Let ρ denote the density of the liquid and $\underline{v}$ the velocity of the liquid. [$\underline{v} = \nabla\phi$]. Let us denote the kinetic energy of the liquid for small motion by T. Then

$$T = \frac{1}{2} \rho \int_\Omega |\underline{v}|^2 = \frac{1}{2} \rho \int_\Omega |\underline{\nabla}\phi|^2$$

Let us denote by u the small variation of the potential energy of the capillary surface when it is displaced from its equilibrium configuration. The expression for u is given by the following :

$$U = \frac{\sigma}{2} \int_\Gamma [b\circ\eta^2 + \underline{\nabla}_\Gamma (\eta,\eta)] + \frac{\sigma}{2} \int_{\partial\Gamma} X\eta^2$$

where η is the normal displacement of Γ [i.e. $u = \frac{\partial \eta}{\partial t}$]. $\underset{\sim}{\nabla}_{\Gamma}$ (...) is the Beltrami first differential operator [Myskis[1987]].

Since Γ is the manifold corresponding to the solution of the capillary surface problem, it is known that under certain conditions it is sufficiently smooth. In fact it is $C^{2+\alpha}$ [$o \le \alpha \le 1$] when the domain of definition of the liquid is convex and when its boundary $\in C^{2+\alpha}$ [Mittelmann [1977], Finn [1986b]]. Under these circumstances the Beltrami Operator is well defined. In our work we assume such smoothness on the capillary surface.

$$bo = \frac{\rho}{\sigma} \frac{\partial \pi}{\partial \underset{\sim}{n}} - k_1^2 - k_2^2 \qquad \text{on } \Gamma$$

Here π is the gravitational potential and k_1, k_2 are the principal curvatures at the point.

$$\chi = \frac{k \cos \theta_c - \bar{k}}{\sin \theta_c} \qquad \text{on } \partial\Gamma$$

Here θ_c is the angle of contact assumed to be constant along $\partial\Gamma$. k and $\bar{k}$ are the curvatures of the normal cross-sections of the surfaces Γ and Σ along the tangent directions $(\underset{\sim}{e}, \underset{\sim}{e}_1)$ of the capillary surface and the vessel surface respectively at the point of contact. For further details regarding the derivation of the potential energy expression one may refer to Myskis [1987]. Considering the variation in T we obtain

$$\delta T = \rho \int_{\Omega} \underset{\sim}{\nabla} \phi . \underset{\sim}{\nabla} \delta\phi = \rho \left(\int_{\Gamma} \frac{\phi \partial \delta\phi}{\partial \underset{\sim}{n}} + \int_{\Sigma} \phi \delta\eta \right) = \rho \int_{\Gamma} \phi \delta\eta \tag{5}$$

where $\dot{\eta} = \frac{\partial \eta}{\partial t}$. In deriving (5) we have made use of Green's formula along with the boundary condition (2).

Also for the variation in $\mathcal{U}$ we have

$$\delta \mathcal{U} = \sigma \int_{\Gamma} (bo\eta - \Delta_{\Gamma}\eta)\ \delta\eta + \sigma \int_{\partial\Gamma} \left[\frac{\partial \eta}{\partial \underline{e}} + X\eta \right] \delta\eta \tag{6}$$

While deriving (6) we have used the Green's formula in a manifold [Myskis[1987]].

According to Hamilton's variational principle we have

$$\delta \int_{t_1}^{t_2} (T-U) = 0 \tag{7}$$

Of course we assume that there are not external forces acting on the system. The $\delta\phi$ and $\delta\eta$ are required to satisfy the isochronism condition. i.e.

$$\delta\phi|_{t=t_1} = \delta\phi|_{t=t_2} = 0 \quad \text{and} \quad \delta\eta|_{t=t_1} = \delta\eta|_{t=t_2} = 0$$

Substituting (5) and (6) in (7) we obtain after integrating by parts the following equation

$$\int_{t_1}^{t_2} \left[\int_{\Gamma} [\rho\frac{\partial \phi}{\partial t} - \sigma(bo\eta - \nabla_{\Gamma}\eta)]\delta\eta - \sigma \int_{\partial\Gamma} (\frac{\partial \eta}{\partial \underline{e}} + X\eta)\delta\eta \right] = 0 \tag{8}$$

Including the volume constraint $\int_{\Gamma}\delta\eta = 0$ with the help of a Langragian multiplier $\psi(t)$ we obtain the following modified form of (8).

$$\int_{t_1}^{t_2} \left[\int_{\Gamma} \left[-\rho \frac{\partial \phi}{\partial t} - \sigma (b o \eta - \nabla_{\Gamma} \eta) - \psi(t) \right] \delta\eta - \sigma \int_{\partial\Gamma} \left(\frac{\partial \eta}{\partial \underline{e}} + X\eta \right) \delta\eta \right] = 0 \tag{9}$$

Hence we obtain the following Euler equation from (9) along with the boundary condition on $\partial\Gamma$.

$$-\rho \frac{\partial \phi}{\partial t} - \sigma \, (b o \eta - \nabla_{\Gamma} \eta) - \psi = 0 \text{ in } \Gamma \tag{10}$$

$$\frac{\partial \eta}{\partial \underline{e}} + X\eta = 0 \text{ on } \partial\Gamma \tag{11}$$

We divide (10) by σ and differentiating (10) and (11) w.r.t time variable t we obtain the following equation.

$$b o u - \Delta_{\Gamma} \dot{\eta} = -\left(\frac{\rho}{\sigma}\right) \frac{\partial^2 \phi}{\partial t^2} \text{ in } \Gamma \tag{12}$$

$$\frac{\partial \eta}{\partial \underline{e}} (\dot{\eta}) + X\dot{\eta} = 0 \text{ on } \partial\Gamma \tag{13}$$

We now assume the harmonic vibrations with frequency $\sqrt{\lambda}$ and substituting u in place of $\dot{\eta}$ (12) and (12) we obtain the following equations :

$$b o u - \Delta_{\Gamma} u = \tilde{\lambda} \left(\frac{\rho}{\sigma}\right) \phi \text{ in } \Gamma \tag{14}$$

$$\frac{\partial u}{\partial \underline{e}} + Xu = 0 \text{ on } \partial\Gamma \tag{15}$$

[where $\tilde{\lambda} = \left(\frac{\rho}{\sigma}\right) \lambda$]

along with the volume constraint condition $\int_{\Gamma} u = 0$. Of course it is implied that ϕ has to satisfy (1) and the boundary condition (2).

Appendix II

In this section we state and prove a theorem which has been used quite regularly in chapter VI.

Theorem A.II.1 : Let $u \in L^2(Y^*)$. If we extend it periodically in $\mathbb{R}^N$ we have

$$\tilde{E}u(\underline{x}/\epsilon) \to m_Y (\tilde{E}u) \text{ in } L^2(\Omega) \qquad (1)$$

[where $\tilde{E}$ is any extension operator]

Proof :

Let $u^\epsilon(\underline{x}) = u(\underline{x}/\epsilon)$ $\forall$ $\underline{x} \in (\Omega)$. It is clear that $\tilde{E}u^\epsilon$ is bounded in $L^2(\Omega)$ as $\epsilon \longrightarrow 0$. Let us choose any $v \in D(\Omega)$. Then we have

$$\int_\Omega \tilde{E}u^\epsilon(\underline{x}) (v - v^*_\epsilon) \underset{\epsilon \to 0}{\longrightarrow} 0$$

[where v^*_ϵ is a function which assumes in each ϵY period a constant value equal to v at the centre of the period]

$$\int_\Omega \tilde{E}u^\epsilon v^*_\epsilon = m_Y (\tilde{E}u) \int_\Omega v^*_\epsilon \underset{\epsilon \to 0}{\longrightarrow} m_Y (\tilde{E}u) \int_\Omega v$$

$$\to \int_\Omega \tilde{E}u^\epsilon(\underline{x})\, v(\underline{x}) \underset{\epsilon \to 0}{\longrightarrow} m_Y (\tilde{E}u) \int_\Omega v(x)$$

Thus (1) is proved.

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